RECOVERING THE GOOD COMPONENT OF THE HILBERT SCHEME

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Abstract. We give an explicit construction, for a flat map $X \to S$ of algebraic spaces, of an ideal in the n'th symmetric product of X over S. Blowing up this ideal is then shown to be isomorphic to the schematic closure in the Hilbert scheme of length n subschemes of the locus of n distinct points. This generalises Haiman's corresponding result ([13]) for the affine complex plane. However, our construction of the ideal is very different from that of Haiman, using the formalism of divided powers rather than representation theory.

In the non-flat case we obtain a similar result by replacing the n'th symmetric product by the n'th divided power product.

The Hilbert scheme, $\mathrm{Hilb}^n_{X/S}$, of length n subschemes of a scheme X over some S is in general not smooth even if $X \to S$ itself is smooth. Even worse, it may not even be (relatively) irreducible. In the case of the affine plane over the complex numbers (where the Hilbert scheme is smooth and irreducible) Haiman (cf., [13]) realised the Hilbert scheme as the blow-up of a very specific ideal of the n'th symmetric product of the affine plane. It is the purpose of this article to generalise Haiman's construction. As the Hilbert scheme in general is not irreducible while the symmetric product is (for a smooth geometrically irreducible scheme over a field say) it does not seem reasonable to hope to obtain a Haiman like description of all of $\mathrm{Hilb}^n_{X/S}$ and indeed we will only get a description of the schematic closure of the open subscheme of n distinct points. With this modification we get a general result which seems very close to that of Haiman. The main difference from the arguments of Haiman is that we need to define the ideal that we want to blow up in a general situation and Haiman's construction seems to be too closely tied to the 2-dimensional affine space in characteristic zero.

As a bonus we get that our constructions work very generally. We have thus tried to present our results in a generality that should cover reasonable applications (encouragement from the referee has made us make it more general than we did in a previous version of this article).

There are some rather immediate consequences of this generality. The first one is that we have to work with algebraic spaces instead

2000 *Mathematics Subject Classification.* 14C05.

Key words and phrases. Hilbert scheme, symmetric product, divided powers, good component.

of schemes as otherwise the Hilbert scheme (as well as the symmetric product) may not exist. A second consequence is that we find ourselves in a situation where existing references do not ensure the existence of $\mathrm{Hilb}^n_{X/S}$ and we give an existence proof in the generality required by us (which is a rather easy patching argument to reduce it to known cases).

It turns out that the key to constructing the ideal to blow up is to use the formalism of divided powers. Recall that if A is a commutative ring and F a flat A-algebra, then the subring of $\mathfrak{S}_n$-invariants of $F^{\otimes_A n}$ is isomorphic to the n'th divided power algebra $\Gamma^n_A(F)$ (through the map that takes $\gamma^n(r)$ to $r^{\otimes n}$).

Using the fact that $\Gamma^n(F)$ is the degree n component of the divided power algebra $\Gamma^*(F)$ we can define an ideal in $\Gamma^n(F)$ (this graded component of the divided power algebra becomes an algebra using the multiplication of F) which is our candidate to be blown up. Note that in the definition of this ideal we are using in an essential way the multiplication in the divided power algebra $\Gamma^*(F)$ forcing us to carefully distinguish between the multiplication in this graded algebra and the multiplication of its graded component $\Gamma^n(F)$ induced by the multiplication on F. On the upside it is exactly this interplay that allows us to define, in a generality outside of Haiman's case, the ideal. Furthermore, the excellent formal properties of $\Gamma^n(F)$ allows us to *define* an analogue of the symmetric product of $\mathrm{Spec}(F) \to \mathrm{Spec}(A)$ as $\mathrm{Spec}(\Gamma^n(F)) \to \mathrm{Spec}(A)$ in the case when $A \to F$ is not flat. This makes our arguments go through without problems in the case when $\mathrm{Spec}(F) \to \mathrm{Spec}(A)$ is not necessarily flat. (We also need to extend the construction of $\mathrm{Spec}(\Gamma^n(F))$ to the non-affine case; the gluing argument needed to make this extension uses results of David Rydh, [20].)

In more detail this paper has the following structure: We start with some preliminaries on divided powers and recall of the Grothendieck-Deligne norm map. The main technical result is to be found in Sections 5 and 6. There we first find a (local) formula for the multiplication of the tautological rank n-algebra over the configuration space of n distinct points of X. We then note that this formula makes sense over the blow-up of a certain ideal in the full symmetric product. This gives us a family of length n subschemes of X over this blow-up and hence a map of it to the Hilbert scheme. Once having constructed it, it is quite easy to show that it gives an isomorphism of the blow-up to the schematic closure of the subspace of n distinct points of the Hilbert scheme. The proof first does this in the case $X \to S$ is affine and then discusses the patching (and limit arguments) needed to extend it to the more general case.

We finish by tying some loose ends. First we generalise the result of Fogarty on the smoothness of $\mathrm{Hilb}^n_{X/S}$ for $X \to S$ smooth of relative dimension 2 removing the conditions on the base S needed by Fogarty.

Finally, we discuss how one can, under suitable conditions, embed the blow-up in a Grassmannian as Haiman does.

0.1. **Acknowledgments.** First we would like to give our thanks to the referee of an earlier version of this article who insisted on putting our results in the most general context and who also sketched how to make such an extension as well as pointing out inaccuracies in the article.

We have received a lot of inspiration with conversations with our colleagues. In particular the second author would like to acknowledge input from Steve Kleiman, Ezra Miller and Dan Laksov.

1. Divided powers and norm

In this section we first recall some properties for the ring of divided powers. The standard reference is Roby [18] and [19], but see also [5] and [9]. Algebras in this note are commutative.

1.1. **The ring of divided powers.** Let A be a commutative ring and M an A-module. The ring of divided powers $\Gamma_A M$ is constructed as follows. We consider the polynomial ring over $A[\gamma^n(x)]_{(n,x)\in \mathbf{N}\times M}$, where the variables $\gamma^n(x)$ are indexed by the set $\mathbf{N}\times M$, where $\mathbf{N}$ is the set of non-negative integers. Then the ring $\Gamma_A M$ is obtained by dividing out the polynomial ring by the following relations

$$\gamma^0(x) - 1 \tag{1.1.1}$$

$$\gamma^n(\lambda x) - \lambda^n \gamma^n(x) \tag{1.1.2}$$

$$\gamma^n(x+y) - \sum_{j=0}^{n} \gamma^j(x)\gamma^{n-j}(y) \tag{1.1.3}$$

$$\gamma^n(x)\gamma^m(x) - \binom{n+m}{n}\gamma^{n+m}(x) \tag{1.1.4}$$

for all integers $m, n \in \mathbf{N}$, all $x, y \in M$, and all $\lambda \in A$. The residue class of the variable $\gamma^n(x)$ in $\Gamma_A M$ we denote by $\gamma^n_M(x)$, or simply $\gamma^n(x)$ if no confusion is likely to occur. The ring $\Gamma_A M$ is graded where $\gamma^n(x)$ has degree n, and with respect to this grading we write $\Gamma_A M = \oplus_{n\geq 0}\Gamma^n_A M$.

1.2. **Polynomial laws.** Let A be a ring, and let M and N be two fixed A-modules. Assume that we for each A-algebra B have a map *of sets* $g_B \colon M\otimes_A B \longrightarrow N\otimes_A B$ such that for any A-algebra homomorphism $u \colon B \longrightarrow B'$ the following diagram is commutative

$$\begin{tikzcd} M\otimes_A B \arrow[r, "g_B"] \arrow[d] & N\otimes_A B \arrow[d] \\ M\otimes_A B' \arrow[r, "g_{B'}"] & N\otimes_A B', \end{tikzcd}$$

where the vertical maps are the canonical homomorphisms. Such a collection of maps is called a polynomial law from M to N, and we denote the polynomial law with $\{g\}\colon M \longrightarrow N$.

Definition 1.3 (Norms)**.** Let A be a ring, M and N two A-modules.

(1) A polynomial law $\{g\}\colon M \longrightarrow N$ is *homogeneous of degree* n if for any A-algebra B we have that $g_B(bx) = b^n g_B(x)$, for any $x \in M \otimes_A B$ and any $b \in B$.

(2) A polynomial law $\{g\}\colon F \longrightarrow E$ between two A-algebras F and E, is *multiplicative* if $g_B(xy) = g_B(x)g_B(y)$ for any x and y in $F \otimes_A B$, for any A-algebra B. Furthermore, we require that $g_B(1) = 1$.

A norm (of degree n) from an A-algebra F to an A-algebra E is a homogeneous multiplicative polynomial law of degree n.

1.4. **Universal norms.** Let n be a non-negative integer. For any A-algebra B we have that $\Gamma^n_A(M) \otimes_A B$ is canonically identified with $\Gamma^n_B(M \otimes_A B)$. It follows that we have a polynomial law $\{\gamma^n\}\colon M \longrightarrow \Gamma^n_A M$ and by (1.1.2) the law is homogeneous of degree n. The polynomial law $\{\gamma^n\}\colon M \longrightarrow \Gamma^n_A M$ is *universal* in the sense that the assignment $u \mapsto \{u \circ \gamma^n\}$ gives a bijection between the A-module homomorphisms $u\colon \Gamma^n_A M \longrightarrow N$ and the set of polynomial laws of degree n from M to N.

Furthermore, if F is an A-algebra then $\Gamma^n_A F$ is an A-algebra and then the polynomial law $\{\gamma^n\}\colon F \longrightarrow \Gamma^n_A F$ is the universal *norm* of degree n ([19, Thm. p. 871], [9, 2.4.2, p. 11]). "Universal" here means in the sense as described above, but for A-algebra homomorphisms from $\Gamma^n_A F$.

1.5. **The different products.** The product structure on $\Gamma_A F$ we refer to as the external structure. We will denote the external product with $*$ in order to distinguish the external product from the product structure on each graded component $\Gamma^n_A F$ defined in the previous section. (Note that our convention is the reverse of the one used in [9].)

1.6. Let p a fixed positive integer, and for each $i = 1, \dots, p$ we assume that we have a sequence of non-negative integers $\{a_{i,j}\}$ $(1 \leq j \leq q_i)$ such that $\sum_{j=1}^{q_i} a_{i,j} = n$. Let $I = [1, \dots, q_1] \times \cdots \times [1, \dots, q_p]$, and let $\mathscr{B}\{a_{i,j}\} \subset \times_I \mathbf{N}$ be the subset of integers $b = \{b_{i_1,\dots,i_p}\}_{i_1,i_2,\dots,i_p \in I}$ such that their sum $|b| := \sum_{i_1,\dots,i_p \in I} b_{i_1,\dots,i_p} = n$ and such that

$$\sum b_{i_1,\dots,i_{r-1},s,i_{r+1},\dots,i_p} = a_{r,s}, \tag{1.6.1}$$

for all r, s.

Proposition 1.7. *Let $q_1, \dots, q_p$ be p positive integers, and let $\{a_{i,j}\}$ be any sequence of non-negative integers such that $\sum_{j=1}^{q_i} a_{i,j} = n$ for each*

$i = 1, \dots, p$. *Then we have for any elements* $\{x_{i,j}\}$ *in the* A*-algebra* F *that in* $\Gamma^n_A F$ *the following identity holds*

$$
\begin{aligned}
&\big(\prod_{j=1}^{q_1} *\gamma^{a_{1,j}}(x_{1,j})\big) \cdots \big(\prod_{j=1}^{q_p} *\gamma^{a_{p,j}}(x_{p,j})\big) \\
&= \sum_{\mathscr{B}\{a_{i,j}\}} \prod_{i_1,\dots,i_p \in I} *\gamma^{b_{i_1,\dots,i_p}}(x_{1,i_1}x_{2,i_2}\cdots x_{p,i_p}).
\end{aligned}
$$

Proof. For each $i = 1, \dots, p$ we consider the linear polynomials

$$L_i = x_{i,1}T_{i,1} + x_{i,2}T_{i,2} + \cdots + x_{i,q_i}T_{i,q_i}$$

in the variables $\{T_{i,j}\}(1 \leq j \leq q_i)$. Let us denote by $A[T]$ the polynomial ring in the variables $\{T_{i,j}\}$ over A, and let $F[T] = A[T] \otimes_A F$. By iterating the formula (1.1.3) and using the fact that the variables are scalars over $A[T]$ combined with (1.1.2) we achieve the following expression

$$\gamma^n(L_i) = \sum_{|\alpha_i|=n} \big(\prod_{j=1}^{q_i} *\gamma^{\alpha_{i,j}}(x_{i,j})\big)T_{i,1}^{\alpha_{i,1}} \cdots T_{i,q_i}^{\alpha_{i,q_i}} \qquad \in \Gamma^n_{A[T]}F[T],$$

where we have abbreviated $\alpha_{i,1} + \cdots + \alpha_{i,q_i} = |\alpha_i|$. We therefore get

$$\gamma^n(L_1)\cdots\gamma^n(L_p) = \prod_{i=1}^{p} \big(\sum_{|\alpha_i|=n} \big(\prod_{j=1}^{q_i} *\gamma^{\alpha_{i,j}}(x_{i,j})\big)T_{i,1}^{\alpha_{i,1}} \cdots T_{i,p_i}^{\alpha_{i,q_i}}\big). \tag{1.7.1}$$

On the other hand we have $\gamma^n(L_1)\cdots\gamma^n(L_p) = \gamma^n(L_1 \cdots L_p)$, thus the identity (1.7.1) above also equals

$$\gamma^n(L_1 \cdots L_p) = \gamma^n\big(\sum_{i_1,\dots,i_p \in I} x_{1,i_1}x_{2,i_2}\cdots x_{p,i_p}T_{1,i_1}T_{2,i_2}\cdots T_{p,i_p}\big), \tag{1.7.2}$$

with $I = [1, \dots, q_1] \times \cdots \times [1, \dots, q_p]$. We then finally iterate the right side of the expression (1.7.2) by the formulas (1.1.2) and (1.1.3), and obtain that (1.7.2) can be written as

(1.7.3)
$$\sum_{|b|=n} \big(\prod_{i_1,\dots,i_p \in I} *\gamma^{b_{i_1,\dots,i_p}}(x_{1,i_1}x_{2,i_2}\cdots x_{p,i_p})T_{1,i_1}^{b_{1,i_1,\dots,p,i_p}} \cdots T_{p,i_p}^{b_{i_1,\dots,i_p}}\big).$$

The sum is to be taken over all integers $b = \{b_{i_1,\dots,i_p}\}$ in $\times_I \mathbf{N}$ such that $|b| = n$. We now compare the coefficients of the polynomial (1.7.1) with the polynomial in (1.7.3). The coefficient of

$$T_{1,1}^{a_{1,1}} \cdots T_{1,q_1}^{a_{1,q_1}} \cdots T_{p,1}^{a_{p,1}} \cdots T_{p,q_p}^{a_{p,q_p}}$$

yields the result. □

Remark 1.8. The proposition is a generalization of the case when $p = 2$ that can be found in ([9, Formula 2.4.2]). The trick we used in the proof is to add variables and then iterate the defining equations for the ring

of divided powers. The technique was communicated to us by Dan Laksov.

As an example of statement in the proposition we give here an identity that we will use later.

Lemma 1.9. *Let* $x_1, \dots, x_n$ *and* f *be elements in an* A*-algebra* F. *Then we have that* $\gamma^1(x_1 f^n) * \gamma^1(x_2) * \cdots * \gamma^1(x_n)$ *equals*

$$\sum_{c=1}^{n} (-1)^{c+1} (\gamma^c(f) * \gamma^{n-c}(1)) \cdot (\gamma^1(x_1 f^{n-c}) * \gamma^1(x_2) * \cdots * \gamma^1(x_n)).$$

Proof. For each $c = 1, \dots, n$ the above proposition identifies the product $\gamma^c(z_{1,1}) * \gamma^{n-c}(z_{1,2}) \cdot \gamma^1(z_{2,1}) * \cdots * \gamma^1(z_{2,n})$ with

$$\sum_{\mathscr{B}\{a^c_{i,j}\}} \prod * \gamma^{b_{i_1,i_2}}(z_{1,i_1} z_{2,i_2}), \tag{1.9.1}$$

where the sequence $\{a^c_{i,j}\}$ is $a^c_{1,1} = c, a^c_{2,1} = n - c$, and $a^c_{2,j} = 1$ for $j = 1, \dots, n$. Let $J = [1,2] \times [2, \dots, n]$, and define $\mathscr{C}^c \subset \times_J \{0,1\}$ as

$$\mathscr{C}^c = \{\{b_{j_1,j_2}\} \mid \sum_{j=2}^{n} b_{1,j} = c, \sum_{j=2}^{n} b_{2,j} = n - c - 1, b_{1,j} + b_{2,j} = 1\}.$$

It is then clear that the indexing set $\mathscr{B}\{a^c_{i,j}\}$ is the disjoint union

$$\mathscr{B}\{a^c_{i,j}\} = \{b_{1,1} = 1, b_{2,1} = 0\} \times \mathscr{C}^{c-1} \sqcup \{b_{1,1} = 0, b_{2,1} = 1\} \times \mathscr{C}^c.$$

We then have that the expression (1.9.1) can be written as

$$\sum_{\mathscr{C}^{c-1}} \gamma^1(z_{1,1} z_{2,1}) * \prod \gamma^{b_{j_1,j_2}}(z_{1,j_1} z_{2,j_2}) + \sum_{\mathscr{C}^c} \gamma^1(z_{1,2} z_{2,1}) * \prod \gamma^{b_{j_1,j_2}}(z_{1,j_1} z_{2,j_2}).$$

Assume now that we have, for each $c = 1, \dots, n$, a collection of elements $\{z^c_{i,j}\}$ with $(i,j) \in [1,2] \times [1, \dots, n]$ such that $z^c_{1,2} z^c_{2,1} = z^{c+1}_{1,1} z^{c+1}_{2,1}$ for $c = 1, \dots, n-1$, and that $z^c_{1,j_1} z^c_{2,j_2} = z^{c+1}_{1,j_1} z^{c+1}_{2,j_2}$ for $(j_1, j_2) \in [1,2] \times [2, \dots, n]$. Using the splitting of $\mathscr{B}\{a^c_{i,j}\}$ described above we have that the alternating sum

$$\sum_{c=1}^{n} (-1)^{c+1} \sum_{\mathscr{B}\{a^c_{i,j}\}} \prod * \gamma^{b_{i_1,i_2}}(z^c_{1,i_1} z^c_{2,i_2})$$

is a telescoping sum. We note that $\mathscr{C}^n$ is the empty set, and that $\mathscr{C}^0$ is the singleton set $\mathscr{C}^0 = \{\{b_{1,j} = 0, b_{2,j} = 1\} \mid j = 2, \dots, n\}$. Consequently the telescoping sum collapses to

$$\gamma^1(z^1_{1,1} z^1_{2,1}) * \gamma^1(z^1_{1,2} z^1_{2,2}) * \cdots * \gamma^1(z^1_{1,2} z^1_{2,n}).$$

The lemma is the special situation with $z^c_{1,1} = f, z^c_{1,2} = 1, z^c_{2,1} = x_1 f^{n-c}$ and $z^c_{2,j} = x_j$ for $j = 2, \dots, n$, $c = 1, \dots, n$. $\square$

1.10. **The canonical homomorphism.** An important norm is the following. Let E be an A-algebra that is free of finite rank n as an A-module. For any A-algebra B we have the determinant map $d_B\colon E\otimes_A B \longrightarrow B$ sending $x \in E \otimes_A B$ to the determinant of the B-linear endomorphism $e \mapsto ex$ on $E \otimes_A B$. It is clear that the determinant maps give a multiplicative polynomial law $\{d\}\colon E \longrightarrow A$, homogeneous of degree $n = \mathrm{rank}_A E$. By the universal properties (1.4) of $\Gamma^n_A E$ we then have an A-algebra homomorphism

$$\sigma_E\colon \Gamma^n_A E \longrightarrow A, \tag{1.10.1}$$

such that $\sigma_E(\gamma^n(x)) = \det(e \mapsto ex)$ for all $x \in E$. In fact we get a homomorphism $\sigma_E\colon \Gamma^n_A E \longrightarrow A$ even if E is only locally free of finite rank n. We call σ_E the canonical homomorphism ([7, Section 6.3, p.180], [14, Section 1.4, p.13]).

Proposition 1.11. *Let E be an A-algebra such that E is free of finite rank n as an A-module. For any element $x \in E$ the characteristic polynomial* $\det(\Lambda - x) \in A[\Lambda]$ *of the endomorphism $e \mapsto ex$ on E is* $\Lambda^n + \sum_{j=1}^n (-1)^j \Lambda^{n-j}\sigma_E(\gamma^j(x) * \gamma^{n-j}(1))$. *In particular we have*

$$\mathrm{Trace}(e \mapsto ex) = \sigma_E(\gamma^1(x) * \gamma^{n-1}(1)).$$

Proof. Let Λ be an independent variable over A, and write $E[\Lambda] = E\otimes_A A[\Lambda]$. By the defining property of the canonical homomorphism $\sigma_{E[\Lambda]}$ we have that the characteristic polynomial $\det(\Lambda - x) = \sigma_{E[\Lambda]}(\gamma^n(\Lambda - x))$. We now use the defining relations (1.1.2) and (1.1.3) in the $A[\Lambda]$-algebra $\Gamma^n_{A[\Lambda]}E[\Lambda]$ and obtain

$$\begin{aligned}\gamma^n(\Lambda - x) &= \sum_{j=0}^n (-1)^j \gamma^j(x) * \gamma^{n-j}(\Lambda)\\ &= \sum_{j=0}^n (-1)^j \Lambda^{n-j}\gamma^j(x) * \gamma^{n-j}(1).\end{aligned}$$

We have that $\Gamma^n_A(R)\otimes_A B = \Gamma^n_B(R\otimes_A B)$ and that $\sigma_{E[\Lambda]} = \sigma_E \otimes \mathrm{id}_{A[\lambda]}$. Consequently $\sigma_{E[\Lambda]}$ acts trivially on the variable Λ and that the action otherwise is as σ_E. Thus we obtain that $\sigma_E(\gamma^j(x) * \gamma^{n-j}(1))$ in A is the j'th coefficient of the characteristic polynomial of $e \mapsto ex$ which proves the claim. □

2. Discriminant and ideal of norms

In this section we define the important ideal of norms and show their connection with discriminants.

Definition 2.1. Let F be an A-algebra. For any $2n$-elements $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_n$ in F we define $\delta(x, y) \in \Gamma^n_A F$ as the

element

$$\delta(x,y) := \det *[\gamma^1(x_iy_j)]_{1\le i,j\le n} = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{|\sigma|}\gamma^1(x_1y_{\sigma(1)})*\cdots*\gamma^1(x_ny_{\sigma(n)}).$$

Remark 2.2. Note that for each element $z \in F$ the element $\gamma^1(z)$ is in $\Gamma^1_A F = F$, but the product $\gamma^1(z_1) * \cdots * \gamma^1(z_n)$ is in $\Gamma^n_A F$.

Lemma 2.3. *Let $x_1,\dots,x_n$ and $y_1,\dots,y_n$ be $2n$-elements in F. Then we have*

$$\det *[\gamma^1(x_iy_j)]_{1\le i,j\le n} = \det[\gamma^1(x_iy_j) * \gamma^{n-1}(1)]_{1\le i,j\le n}.$$

Proof. We have that the determinant of $[\gamma^1(x_iy_j) * \gamma^{n-1}(1)]$ is

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{|\sigma|}\big(\gamma^1(x_1y_{\sigma(1)}) * \gamma^{n-1}(1)\big)\cdots\big(\gamma^1(x_ny_{\sigma(n)}) * \gamma^{n-1}(1)\big). \tag{2.3.1}$$

We will now use Proposition (1.7) to expand the product in (2.3.1). For that purpose we denote $I = [1,2]^n$, $a_{i,1} = 1$ and $a_{i,2} = n-1$, and we write $X^\sigma_{i,1} = x_iy_{\sigma(i)}$ and $X^\sigma_{i,2} = 1$ for all $i = 1,\dots,n$. Using Proposition (1.7), we then rewrite the expression (2.3.1) as

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{|\sigma|} \sum_{b\in\mathscr{B}\{a_{i,j}\}} \prod_{i_1,\dots,i_n\in I} *\gamma^{b_{i_1,\dots,i_n}}(X^\sigma_{1,i_1}\dots X^\sigma_{n,i_n}). \tag{2.3.2}$$

Consider now the element $\mathbf{b} \in \times_I \mathbf{N}$ that is zero on all components except the components indexed by $i_1,\dots,i_n$ where all indices but one $i_r = 1$ (and thus the other indices are $= 2$), and let the values of $\mathbf{b}$ at those components all equal 1. That is

$$b_{1,2\dots,2} = b_{2,1,2,\dots,2} = \cdots = b_{2,\dots,2,1} = 1.$$

We then have that $\mathbf{b}$ satisfies the equations (1.6.1), hence $\mathbf{b} \in \mathscr{B}$. We see that $\delta(x,y)$ occurs as a summand in (2.3.2) corresponding to having $\mathbf{b}$ fixed. We can therefore write (2.3.2) as

$$\delta(x,y) + \sum_{\sigma\in\mathfrak{S}_n} (-1)^{|\sigma|} \sum_{b\in\mathscr{B}\{a_{i,j}\}\setminus\mathbf{b}} \prod_{i_1,\dots,i_n\in I} *\gamma^{b_{i_1,\dots,i_n}}(X^\sigma_{1,i_1}\dots X^\sigma_{n,i_n}) \tag{2.3.3}$$

We need to show that the right hand side of (2.3.3) is zero. By using the equations described by (1.6.1) one gets that if $b_{i_1,\dots,i_n}$ is a non-zero integer with at least one index $i_r = 1$, then because $a_{r,1} = 1$ we must have the value $b_{i_1,\dots,i_r=1,\dots,i_n} = 1$. Furthermore it follows easily that if $b \in \mathscr{B}\{a_{i,j}\}$ where one component $b_{i_1,\dots,i_n} = 1$ with only one index $i_r = 1$, then $b = \mathbf{b}$ defined above. Thus a non-zero component of an element $b \in \mathscr{B}\{a_{i,j}\} \setminus \mathbf{b}$ has either all indices $i_1 = \cdots = i_n = 2$, or at least two indices $i_r = i_{r'} = 1$.

The first alternative with $i_1 = \cdots = i_n = 2$ gives that the only non-zero component of b is $b_{2,\dots,2}$, and that alternative is impossible as it should equal $n-1$ by (1.6.1), but also it should equal $|b| = n$, being an element of $\mathscr{B}\{a_{i,j}\}$.

Let us now rule out the other alternative, with at least two indices $i_r = i_{r'} = 1$ of an element $b \in \mathscr{B}\{a_{i,j}\}$. The equations (1.6.1) then imply that all the other non-zero components must have the indices $i_r = i_{r'} = 2$. From that we deduce that the product

$$\prod_{i_1,\dots,i_n \in I} *\gamma^{b_{i_1,\dots,i_n}}(X^\sigma_{1,i_1}\dots X^\sigma_{n,i_n})$$

is invariant under the action of ϵ, the operation that permutes the two factors r and r'. Because in one component of the product (corresponding to $i_r = i_{r'} = 1$) we have

$$X^\sigma_{1,i_r}X^\sigma_{1,i_{r'}} = x_{i,r}y_{\sigma(i_r)}x_{i,r'}y_{\sigma(i'_r)} = X^{\epsilon\sigma}_{1,i_r}X^{\epsilon\sigma}_{1,i'_r}.$$

In the other components of the product we have $X^\sigma_{2,i_r} = X^\sigma_{2,i'_r} = 1$, clearly invariant under permutation. As ϵ has sign -1 it is clear that the above sum is annihilated in the determinant expression. As the above sum was arbitrary we have that the right side of (2.3.3) is zero. $\square$

Lemma 2.4. *Let $x = x_1,\dots,x_n$ and $y = y_1,\dots,y_n$ be $2n$ elements in an A-algebra F. We have $\delta(x,y)^2 = \delta(x,x)\delta(y,y)$.*

Proof. By definition $\delta(x,y)^2$ is the sum

$$\sum_{\sigma,\tau\in\mathfrak{S}_n}(-1)^{|\sigma\tau|}\big(\prod_{i=1}^n *\gamma^1(x_iy_{\sigma(i)})\big)\cdot\big(\prod_{j=1}^n *\gamma^1(x_jy_{\tau(j)})\big).$$

We expand each summand using Proposition (1.7) which gives that $\delta(x,y)^2$ can be written as

$$\sum_{\sigma,\tau\in\mathfrak{S}_n}(-1)^{|\sigma\tau|}\sum_{\rho\in\mathfrak{S}_n}\prod_{i=1}^n *\gamma^1(x_iy_{\sigma(i)}x_{\rho(i)}y_{\tau(\rho(i))}).$$

On the other hand we have that $\delta(x,x)\delta(y,y)$ is

$$\sum_{\rho,\beta\in\mathfrak{S}_n}(-1)^{|\rho\beta|}\big(\prod_{i=1}^n *\gamma^1(x_ix_{\rho(i)})\big)\cdot\big(\prod_{j=1}^n *\gamma^1(y_jy_{\beta(j)})\big).$$

Expanding the products as in Proposition (1.7), and then substituting $\beta = \tau\rho\sigma^{-1}$, proves the lemma. $\square$

Lemma 2.5. *Let $x = x_1,\dots,x_n$ and $y = y_1,\dots,y_n$ be $2n$ elements in an A-algebra F. Let $z_j = \sum_{i=1}^n a_{i,j}x_i$ and $w_j = \sum_{i=1}^n b_{i,j}y_i$ be A-linear combinations of x and y for $j = 1,\dots,n$. Then we have*

$$\delta(z,w) = \det(a_{i,j})\det(b_{i,j})\delta(x,y).$$

Proof. We clearly have the two matrix equations

$$Z = (a_{i,j})X \quad\text{and}\quad W = (b_{i,j})Y, \tag{2.5.1}$$

where X, Y, Z and W are the $(n\times 1)$-matrices with entries x_i, y_i, z_i and w_i ($1 \le i \le n$), respectively.

We have $ax = a\gamma^1(x)$ in $\Gamma^1_A F = F$ for any $a \in A$ and any $x \in F$. We therefore view (2.5.1) as a matrix equation over $\Gamma^1_A F$. We then multiply the matrix Z with the transpose of W and obtain an $(n \times n)$-matrix

$$Z \cdot W^{tr} = (a_{i,j})X \cdot Y^{tr}(b_{i,j}). \tag{2.5.2}$$

As we have $\gamma^1(z) \cdot \gamma^1(w) = \gamma^1(zw)$ in $\Gamma^1_A F$ we read (2.5.2) as

$$(\gamma^1(z_i w_j)) = (a_{i,j})(\gamma^1(x_i y_j))(b_{i,j})^{tr}. \tag{2.5.3}$$

The A-algebra structure on the ring $\Gamma_A F$ is compatible with the A-algebra structure on each of its graded components $\Gamma^m_A F$, and we will therefore view (2.5.2) as an equation of four matrices $Z \cdot W^{tr}$, $(a_{i,j})$, $X \cdot Y^{tr}$ and $(b_{i,j})$ over the A-algebra $\Gamma_A F$. Now, using the usual properties of the determinant we obtain the following identity

$$\det *(\gamma^1(z_i w_j)) = (\det *(a_{i,j})) * (\det *(\gamma^1(x_i y_j))) * (\det *(b_{i,j})).$$

As the entries of $(a_{i,j})$ and the entries of $(b_{i,j})$ are in the ring $A = \Gamma^0_A F$ we get that $\det *(a_{i,j}) = \det(a_{i,j})$, and similarly that $\det *(b_{i,j}) = \det(b_{i,j})$. It then follows that the last displayed equation is what we wanted to prove. □

Definition 2.6 (The ideal of norms)**.** Let n be a fixed integer, and let $V \subseteq F$ be an A-submodule of an A-algebra F. We define $I_V \subseteq \Gamma^n_A F$, the ideal of norms associated to V, as the ideal generated by

$$\delta(x, y) = \det *[\gamma^1(x_i y_j)]_{1 \leq i,j \leq n} \in \Gamma^n_A F$$

for any $2n$-elements $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$ in $V \subseteq F$.

Lemma 2.7. *Let $A \longrightarrow B$ be a homomorphism of rings, and let $V \subseteq F$ be an A-submodule of an A-algebra F. The extension of the ideal I_V by the A-algebra homomorphism $\Gamma^n_A F \longrightarrow \Gamma^n_A(F) \otimes_A B$ equals the ideal I_{V_B}; the ideal of norms associated to the B-submodule $Im(V \otimes_A B \longrightarrow F \otimes_A B)$.*

Proof. Let $x = x_1, \ldots, x_n$ and $y = y_1, \ldots, y_n$ be $2n$-elements in $V \subseteq F$. The element $\delta(x, y)$ in $\Gamma^n_A F$ is then mapped to $\delta(x, y) \otimes 1_B = \det *[\gamma^1_F(x_i y_j) \otimes 1_B]$ in $\Gamma^n_A(F) \otimes_A B$. We have that $\Gamma^n_A(F) \otimes_A B$ is canonically identified with $\Gamma^n_B(F \otimes_A B)$. Hence

$$\det *[\gamma^1_F(x_i y_j) \otimes 1_B]_{1 \leq i,j \leq n} = \det *[\gamma^1_{F \otimes_A B}(x_i y_j \otimes 1_B)]_{1 \leq i,j \leq n}.$$

We then have that the extension $I_V \Gamma^n_B(F \otimes_A B)$ is included in I_{V_B}. As the generators of the B-module V_B are the images of generators of the A-module V, the inclusion $I_{V_B} \subseteq I_V \Gamma^n_B(F \otimes_A B)$ follows from Lemma (2.5). □

Lemma 2.8. *Let $F = A[T_1, \ldots, T_r]$ be the polynomial ring in a finite set of variables, and let $V \subset F$ be the A-module spanned by those monomials whose degree in each of the variables is less than n. Then the ideals of norms associated to V and F are equal; that is $I_V = I_F$.*

Furthermore, if $n!$ is invertible in A then $I_W = I_F$, where $W \subset F$ is the A-module spanned by monomials of degree less than n.

Proof. Given $x_1, \dots, x_n$ and f in F we write $x(c) = x_1 f^c, x_2, \dots, x_n$. For any $y_1, \dots, y_n$ we then obtain from the equality given in Lemma (1.9) that

$$\delta(x(n), y) = \sum_{c=1}^{n} (-1)^{c+1} (\gamma^c(f) * \gamma^{n-c}(1)) \cdot \delta(x(n-c), y).$$

The first assertion of the lemma follows from the above equality. When $n!$ is invertible, the n'th powers of linear forms span the module of degree n monomials, and the above equality then also yields the second assertion. □

2.9. **Discriminant.** Let E be an A-algebra that is free of finite rank n as an A-module. The trace of an A-linear endomorphism of E is an A-linear map $\mathrm{End}_A(E) \longrightarrow A$, that composed with the natural map $E \longrightarrow \mathrm{End}_A(E)$ sends an element $x \in E$ to $\mathrm{tr}(x)$; the trace of the endomorphism $e \mapsto ex$ on E. There is an associated map $E \longrightarrow \mathrm{Hom}_A(E, A)$ taking $y \in E$ to the trace $\mathrm{tr}(xy)$, for any $x \in E$.

The discriminant ideal $D_{E/A} \subseteq A$ is defined (see e.g. [1, p. 124]) as the ideal generated by the determinant of the associated map $E \longrightarrow \mathrm{Hom}(E, A)$.

Proposition 2.10. *Let E be an A-algebra that is free of finite rank n as an A-module. Then the extension of I_V, the ideal of norms associated to $V = E$, by the canonical homomorphism $\sigma_E \colon \Gamma^n_A E \longrightarrow A$ is the discriminant ideal. In particular we have that the extension $\sigma_E(I_V)A = A$ if and only if $\mathrm{Spec}(E) \longrightarrow \mathrm{Spec}(A)$ is étale.*

Proof. By Lemma (2.5) we have that the ideal I_V is generated by the single element $\delta(x, x) = \det *[\gamma^1(x_i x_j)]$, where $x = x_1, \dots, x_n$ is an A-module basis of $E = V$. By Lemma (2.3) we have the identity $\det[\gamma^1(x_i x_j)] = \det[\gamma^1(x_i x_j) * \gamma^{n-1}(1)]$ in $\Gamma^n_A F$. As σ_E is an algebra homomorphism we have

$$\sigma_E \det[\gamma^1(x_i x_j) * \gamma^{n-1}(1)] = \det[\sigma_E(\gamma^1(x_i x_j) * \gamma^{n-1}(1))].$$

By Proposition (1.11) we have $\sigma_E(\gamma^1(e_i e_j) * \gamma^{n-1}(1)) = \mathrm{Trace}(e \mapsto e x_i x_j)$. Thus we have a matrix with entries $\mathrm{Trace}(e \mapsto e x_i x_j)$, and the determinant is then the discriminant. □

3. Connection with symmetric tensors

3.1. **A norm vector.** Let F be an A-algebra, and let $T^n_A F = F \otimes_A \cdots \otimes_A F$ be the tensor product with n-copies of F. We fix the positive integer n, and for any element $x \in F$ we use the following notation

$$x_{[j]} = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1, \qquad (3.1.1)$$

where the x occurs at the j'th component of $T^n_A F$. The group $\mathfrak{S}_n$ of n-letters acts on $T^n_A F$ by permuting the factors. For any n-elements $x = x_1, \dots, x_n$ in F we define the norm vector

$$\nu(x) = \nu(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \in T^n_A F,$$

where the summation runs over the elements σ in the group $\mathfrak{S}_n$. The norm vector $\nu(x)$ is not a symmetric tensor since clearly we have that $\nu(x) = \det(x_{i,[j]})$, where $(x_{i,[j]})$ is the square matrix with coefficients $x_{i,[j]}$ on row i, and column j, with $1 \leq i, j \leq n$. It follows that $\nu(x_{\rho(1)}, \dots, x_{\rho(n)}) = (-1)^{|\rho|} \nu(x_1, \dots, x_n)$ for any $\rho \in \mathfrak{S}_n$, and that

$$\nu(x) = 0 \quad \text{if } x_i = x_j, \text{ with } i \neq j. \tag{3.1.2}$$

3.2. Let $\mathrm{TS}^n_A F$ denote the invariant ring of $\mathrm{T}^n_A F$ by the natural action of the symmetric group $\mathfrak{S}_n$ in n-letters that permutes the factors. We have the map $F \longrightarrow \mathrm{T}^n_A F$ sending $x \mapsto x \otimes \cdots \otimes x$, and it is clear that the map factors through the invariant ring $\mathrm{TS}^n_A F$. The map $F \longrightarrow \mathrm{TS}^n_A F$ determines a norm of degree n, as one readily verifies, hence there exist an A-algebra homomorphism

$$\alpha_n \colon \Gamma^n_A F \longrightarrow \mathrm{TS}^n_A F \tag{3.2.1}$$

such that $\alpha_n(\gamma^n(x)) = x \otimes \cdots \otimes x$, for all $x \in F$.

3.3. **The shuffle product.** When F is an A-algebra that is flat as an A-module, or if $n!$ is invertible in A, then the A-algebra homomorphism α_n (3.2.1) is an isomorphism ([18, IV, §5. Proposition IV.5], [5, Exercise 8(a), AIV. p.89]). In those cases we can identify $\Gamma_A F$ as the graded sub-module

$$\Gamma_A F = \bigoplus_{n \geq 0} TS^n_A F \subseteq \bigoplus_{n \geq 0} \mathrm{T}^n_A F = \mathrm{T}_A F.$$

The external product structure on $\Gamma_A F$ is then identified with the shuffle product on the full tensor algebra $\mathrm{T}_A F$. The shuffle product of an n-tensor $x \otimes \cdots \otimes x$ and an m-tensor $y \otimes \cdots \otimes y$ is the $m+n$-tensor given as the sum of all possible different shuffles of the n copies of x and m copies of y ([5, Exercise 8 (b), AIV. p.89]).

Proposition 3.4. *Let F be an A-algebra, and let $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_n$ be $2n$ elements of F. The A-algebra homomorphism $\alpha_n \colon \Gamma^n_A F \longrightarrow \mathrm{TS}^n_A F$ (3.2.1) is such that*

$$\alpha_n(\delta(x, y)) = \nu(x)\nu(y).$$

Proof. The homomorphism $\alpha_n \colon \Gamma^n_A F \longrightarrow \mathrm{TS}^n_A F$ takes the external product $*$ of $\Gamma_A F$ to the shuffle product of $\mathrm{TS}_A F$. We then have

$$\begin{aligned}\alpha_n(\delta(x,y)) &= \sum_{\sigma\in\mathfrak{S}_n}(-1)^{|\sigma|}\alpha_n(\gamma^1(x_1y_{\sigma(1)})*\cdots*\gamma^1(x_ny_{\sigma(n)}))\\ &= \sum_{\sigma\in\mathfrak{S}_n}(-1)^{|\sigma|}\sum_{\tau\in\mathfrak{S}_n}(x_{\tau(1)}y_{\sigma(\tau(1))})\otimes\cdots\otimes(x_{\tau(n)}y_{\sigma(\tau(n))})\end{aligned}$$

By formally manipulating the expression above we obtain

$$\begin{aligned}&= \sum_{\tau\in\mathfrak{S}_n}(-1)^{|\tau|}\sum_{\sigma\in\mathfrak{S}_n}(-1)^{|\sigma||\tau|}(x_{\tau(1)}y_{\sigma(\tau(1))})\otimes\cdots\otimes(x_{\tau(n)}y_{\sigma(\tau(n))})\\ &= \sum_{\tau\in\mathfrak{S}_n}(-1)^{|\tau|}\sum_{\tau'\in\mathfrak{S}_n}(-1)^{|\tau'|}(x_{\tau(1)}y_{\tau'(1)})\otimes\cdots\otimes(x_{\tau(n)}y_{\tau'(n)})\\ &= \Big(\sum_{\tau\in\mathfrak{S}_n}(-1)^{|\tau|}x_{\tau(1)}\otimes\cdots\otimes x_{\tau(n)}\Big)\Big(\sum_{\tau'\in\mathfrak{S}_n}(-1)^{|\tau'|}y_{\tau'(1)}\otimes\cdots\otimes y_{\tau'(n)}\Big)\\ &= \nu(x)\nu(y),\end{aligned}$$

and we arrive at $\alpha_n(\delta(x,y)) = \nu(x)\nu(y)$ as claimed. □

Corollary 3.5. *Let $\tilde{\alpha} \colon \Gamma^n_A F \longrightarrow \mathrm{T}^n_A F$ denote the composition of the map α_n and the inclusion $\mathrm{TS}^n_A F \subseteq \mathrm{T}^n_A F$. Let $I \subseteq \mathrm{T}^n_A F$ denote the extension of the ideal of norms I_F by $\tilde{\alpha}$, and let $J \subseteq \mathrm{T}^n_A F$ denote the ideal of the diagonals. Then we have $\sqrt{I} = \sqrt{J}$.*

Proof. Let $\varphi\colon \mathrm{T}^n_A F \longrightarrow L$ be a morphism with L a field, and let $\varphi_i \colon F \longrightarrow L$ be the composition of φ and the i'th co-projection $F \longrightarrow \mathrm{T}^n_A F$, where $i = 1, \dots, n$. If φ corresponds to a point in the open complement of the diagonals then all the maps φ_i are different. That is, no $\mathfrak{p}_i = \ker(\varphi_i)$ is contained in another $\mathfrak{p}_j$. Furthermore, since the kernels also are prime ideals there exists, for each i, an element x_i not in $\mathfrak{p}_i$, but where $x_i \in \mathfrak{p}_j$ when $j \neq i$. We then have that $\varphi_j(x_i) = 0$ for $j \neq 0$, and that $\varphi_i(x_i) \neq 0$. Hence there are elements $x_1, \dots, x_n$ in F such that $\det(\varphi_j(x_i)) \neq 0$. Then also the image of $\nu(x_1, \dots, x_n)$ is non-zero in L, and we have that the point φ is in the open complement of the scheme defined by $I \subseteq \mathrm{T}^n_A F$.

Conversely, if φ corresponds to a point on the diagonals then at least two of the maps φ_i are equal. Consequently, for any elements $x_1, \dots, x_n$ in F we have that $\varphi(\nu(x_1, \dots, x_n)) = 0$. It follows that $I \subseteq \ker\varphi$, proving the claim. □

4. Grothendieck-Deligne norm map

In this section we recall the Grothendieck-Deligne norm map following Deligne ([7]), and we discuss briefly the related Hilbert-Chow morphism. Furthermore we define the notion of sufficiently big submodules.

4.1. **The Hilbert functor of n-points.** We fix an A-algebra F, and a positive integer n. We let Hilb^n_F denote the covariant functor from the category of A-algebras to sets, that sends an A-algebra B to the set

$$\mathrm{Hilb}^n_F(B) = \{\text{ideals in } F \otimes_A B \text{ such that the quotient } E \text{ is locally free of rank } n \text{ as a } B\text{-module}\}.$$

4.2. **The Grothendieck-Deligne norm.** If E is an B-valued point of Hilb^n_F we have the sequence

$$F \longrightarrow F \otimes_A B \longrightarrow E, \tag{4.2.1}$$

from where we obtain the A-algebra homomorphisms $\Gamma^n_A F \longrightarrow \Gamma^n_B E$ that sends $\gamma^n(x)$ to $\gamma^n(\bar{x} \otimes 1)$, where $\bar{x} \otimes 1$ is the residue class of $x \otimes 1$ in E. Furthermore, when we compose the homomorphism $\Gamma^n_A F \longrightarrow \Gamma^n_B E$ with the canonical homomorphism $\sigma_E \colon \Gamma^n_B E \longrightarrow B$ we obtain an assignment that is functorial in B; that is we have a morphism of functors

$$\mathrm{n}_F \colon \mathrm{Hilb}^n_F \longrightarrow \mathrm{Hom}_{A\text{-alg}}(\Gamma^n_A F, -). \tag{4.2.2}$$

The natural transformation n_F we call the Grothendieck-Deligne norm map.

Remark 4.3. The Hilbert functor Hilb^n_F can in a natural way be viewed as a contra-variant functor from the category of schemes (over $\mathrm{Spec}(A)$) to sets. In that case the functor Hilb^n_F is representable by a scheme (see e.g. [12]). If $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ we write $\mathrm{n}_X \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ for the morphism that corresponds to the natural transformation (4.2.2).

4.4. **The geometric action.** Let $A = K$ be an algebraically closed field, and let E be a finitely generated Artinian K-algebra. As E is Artinian it is a product of local rings $E = \prod_{i=1}^p E_i$, and we let $\rho_i \colon E \longrightarrow K$ denote the residue class map that factors via E_i. Let $m_i = \dim_K(E_i)$, and let $n = \dim_K(E) = m_1 + \cdots + m_p$. Iversen ([14, Proposition 4.7]) shows that the canonical homomorphism $\sigma_E \colon \Gamma^n_K E = \mathrm{TS}^n_K E \longrightarrow K$ factors via the homomorphism $\rho \colon T^n_K E \longrightarrow K$, where

$$\rho = (\rho_1, \dots, \rho_1, \dots, \rho_p, \dots, \rho_p),$$

and where each factor ρ_i is repeated m_i-times.

4.5. **Hilbert-Chow morphism.** Assume that the base ring $A = K$ is a field, and let $X = \mathrm{Spec}(F)$. Then we can identify $\mathrm{Spec}(\Gamma^n_K F)$ with the symmetric quotient $\mathrm{Sym}^n(X) := \mathrm{Spec}(\mathrm{TS}^n_K F)$. Furthermore we have that the $\mathrm{Spec}(K)$-valued points of Hilb^n_X correspond to closed zero-dimensional subschemes $Z \subseteq X$ of length n. When K is algebraically

closed we have by (4.4) that the Grothendieck-Deligne norm map sends an K-valued point $Z \subseteq X$ to the "associated" zero-dimensional cycle

$$\mathrm{n}_X(Z) = \sum_{P \in |Z|} \dim_K(\mathscr{O}_{Z,P})[P],$$

where the summation runs over the points in the support of Z. Hence we see that the norm morphism n_X has the same effect on geometric points as the Hilbert-Chow morphism. The Hilbert-Chow morphism that appears in [10] and [8] requires that the Hilbert scheme is reduced, whereas the Hilbert-Chow morphism that appears in [16] requires that the Hilbert scheme is (semi-) normal. As the morphism n_X does not require any hypothesis on the source we have chosen to refer to that morphism with a different name; the Grothendieck-Deligne norm map.

Lemma 4.6. *Let $A = K$ be a field of characteristic zero, and let $F = K[T]$ be the polynomial ring in a finite set of variables $T_1, \dots, T_r$. For $n > 0$ the K-algebra $\Gamma^n_K F$ is generated by*

$$\gamma^1(m) * \gamma^{n-1}(1),$$

for monomials $m \in K[T]$ of degree deg$(m) \leq n$.

Proof. The identification $\alpha_n \colon \Gamma^n_K K[T] \longrightarrow \mathrm{TS}^n_K\, K[T]$ identifies, for any $m \in K[T]$, the element $\gamma^1(m) * \gamma^{n-1}(1)$ with the shuffled product of $\alpha_1(m) = m$ and $\alpha_{n-1}(1) = 1 \otimes \cdots \otimes 1$. That is

$$\alpha_n(\gamma^1(m) * \gamma^{n-1}(1)) = m \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes \cdots 1 \otimes m = P(m).$$

By a well-known result of Weyl ([21, II 3]) the invariant ring $\mathrm{TS}^n_K\, F$ is generated by the power sums $P(m)$ of monomials $m \in K[T]$ of degree less or equal to n. □

Definition 4.7 (Sufficiently big modules)**.** Let us fix an A-algebra F. An A-submodule $V \subseteq F$ is n-sufficiently big if the composite B-module homomorphism

$$V \otimes_A B \longrightarrow F \otimes_A B \longrightarrow E$$

is surjective for all A-algebras B, and all B-valued points E of the Hilbert functor Hilb^n_F.

Remark 4.8. Sufficiently big submodules always exist as we can take $V = F$.

Remark 4.9. If V is sufficiently big then we clearly have a morphism of functors

$$\mathrm{Hilb}^n_F \longrightarrow \mathrm{Grass}^n_V$$

from the Hilbert functor of rank n-families, to the Grassmannian of locally free rank n-quotients of V.

Theorem 4.10. *Let F be an A-algebra, n an positive integer and let $V \subseteq F$ be an n-sufficiently big submodule. Then we have for any A-algebra B, and any B-valued point E of Hilb^n_F that the extension of I_V, the ideal of norms associated to V, by the Grothendieck-Deligne norm map $\mathrm{n}_F \colon \Gamma^n_A F \longrightarrow B$ is the discriminant ideal of E over B. That is*

$$\mathrm{n}_F(I_V)B = D_{E/B} \subseteq B.$$

Proof. We first make a reduction to the situation when B is a local ring. For any A-algebra homomorphism $B \longrightarrow B'$ we let $E' = E \otimes_B B'$, and we have the commutative diagram

$$\begin{tikzcd} \Gamma^n_A F \arrow[r] & \Gamma^n_B E \arrow[r, "\sigma_E"] \arrow[d] & B \arrow[d] \\ & \Gamma^n_{B'} E' \arrow[r, "\sigma_{E'}"] & B'. \end{tikzcd}$$

Thus by letting B' be the local ring of a prime ideal in B, and using the fact that discriminant ideals are compatible with base change, we may assume that B is a local ring.

Let B be a local ring with residue field K. As the composition map

$$V \otimes_A K \longrightarrow F \otimes_A K \longrightarrow E \otimes_A K$$

is surjection of K-vector spaces we can find elements $x_1, \dots, x_n$ in V such that the residue classes of $x_1 \otimes \mathrm{id}_K, \dots, x_n \otimes \mathrm{id}_K$ in $E \otimes_A K$ form a K-vector space basis. It then follows from Nakayama's Lemma that the residue classes of $x_1 \otimes \mathrm{id}_B, \dots, x_n \otimes \mathrm{id}_B$ form a B-module basis of $E \otimes_A B = E$. Then we have that the residue class $\bar{y} \otimes \mathrm{id}_B$ for any element $y \in F$, can be written as a B-linear combination of $\bar{x}_1 \otimes \mathrm{id}_B, \dots, \bar{x}_n \otimes \mathrm{id}_B$. In particular we have by Lemma (2.5) that $I_E \subseteq \Gamma^n_B E$ the ideal of norms associated to $V = E$ is generated by the single element

$$\delta(x, x) = \det * [\gamma^1(\bar{x_i x_j} \otimes \mathrm{id}_B)]_{1 \leq i,j \leq n} \in \Gamma^n_B E. \tag{4.10.1}$$

By Lemma (2.7) we have that the extension of $I_V \subseteq \Gamma^n_A F$ by the composite map $\Gamma^n_A F \longrightarrow \Gamma^n_B E$ is exactly I_E. Thus we obtain that the extension of I_V by the Grothendieck-Deligne norm map n_F is the ideal generated by the image of the element (4.10.1) in B. By the definition of n_F we apply the canonical homomorphism $\sigma_E \colon \Gamma^n_B E \longrightarrow B$ to obtain the image of (4.10.1) in B. The result now follows from Proposition (2.10). □

5. Families of distinct points

5.1. **The canonical morphism.** The map $F \longrightarrow F \otimes_A \Gamma^{n-1}_A F$ sending z to $z \otimes \gamma^{n-1}(z)$ determines a norm of degree n. Consequently there is

a unique A-algebra homomorphism $\Gamma^n_A F \longrightarrow F \otimes_A \Gamma^{n-1}_A F$ that takes $\gamma^n(z)$ to $z \otimes \gamma^{n-1}(z)$. Let

$$\pi_n \colon \operatorname{Spec}(F) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(\Gamma^{n-1}_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F) \tag{5.1.1}$$

denote the corresponding morphism of schemes. Furthermore, we let $\Delta \subseteq \operatorname{Spec}(\Gamma^n_A F)$ denote the closed subscheme corresponding to the ideal of norms associated to F.

Proposition 5.2. *Let $U = \operatorname{Spec}(\Gamma^n_A F) \setminus \Delta$ denote the open set where the ideal sheaf of norms vanishes. Then the induced morphism*

$$\pi_{n|} \colon \pi_n^{-1}(U) \longrightarrow U$$

is étale of rank n.

Proof. Let $U_n \subseteq \operatorname{Spec}(\mathrm{T}^n_A F)$ denote the open complement of the diagonals. The group of n-letters, $\mathfrak{S}_n$, acts freely on U_n and the quotient map $U_n \longrightarrow U_n/\mathfrak{S}_n$ is étale of rank $n! = |\mathfrak{S}_n|$. The morphism $\operatorname{Spec}(\alpha_n) \colon \operatorname{Spec}(\mathrm{TS}^n_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F)$ is an isomorphism when restricted to $U_n/\mathfrak{S}_n$ (see e.g. [20, Prop. 4.2.6]). It then follows from Corollary (3.5) that the map $\tilde{\alpha}_n : \operatorname{Spec}(\mathrm{T}^n_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F)$ is an $\mathfrak{S}_n$-torsor over $\operatorname{Spec}(\Gamma^n_A F) \setminus \Delta$. Furthermore, after a faithfully flat base change $A \longrightarrow A'$ we can assume that $\Gamma^n_A(F) \otimes_A A' = \Gamma^n_{A'}(F \otimes_A A')$ is generated by elements of the form $\gamma^n(z)$ ([9, Lemma 2.3.1]). Then clearly the diagram

$$\begin{tikzcd} \operatorname{Spec}(\mathrm{T}^n_A F) \arrow[rr, "\tilde{\alpha}_n"] \arrow[dr, "1\times\tilde{\alpha}_{n-1}"'] & & \operatorname{Spec}(\Gamma^n_A F) \\ & \operatorname{Spec}(F) \times \operatorname{Spec}(\Gamma^{n-1}_A F) \arrow[ur, "\pi_n"'] & \end{tikzcd}$$

is commutative. Over the complement of $\Delta \subseteq \operatorname{Spec}(\Gamma^n_A F)$ we have that $\tilde{\alpha}_n$ is étale of rank $n!$, and $1 \times \tilde{\alpha}_{n-1}$ is étale of rank $(n-1)!$. And consequently π_n is étale of rank n. $\square$

5.3. **Notation.** We have the ordered sequence $x = x_1, \dots, x_n$ of elements in F fixed. Let $U_A(x)$ be $\Gamma^n_A F$ localized at the element $\delta(x,x)$, and consider the induced map

$$U_A(x) \longrightarrow (F \otimes_A \Gamma^{n-1}_A F) \otimes_{\Gamma^n_A F} U_A(x) = M_A(x) \tag{5.3.1}$$

obtained by localization of (5.1.1).

Lemma 5.4. *The images of the elements $x = x_1, \dots, x_n$ by the map $F \longrightarrow F \otimes_A U_A(x) \longrightarrow M_A(x)$ form an $U_A(x)$-module basis for $M_A(x)$.*

Proof. By Proposition (5.2) we have that $M_A(x)$ is Zariski locally free of rank n over $U_A(x)$. To show that $M_A(x)$ is free it suffices to show that the images of $x_1, \dots, x_n$ form a basis locally. Hence we may assume that M is a free U-module, where U is some localization of $U_A(x)$. Let $e = e_1, \dots, e_n$ be a basis of M, and let $q(z)$ denote the image of $z \in F$

in M. There exist scalars $a_{i,j} \in U$ such that $q(x_i) = \sum_{j=1}^n a_{i,j} e_j$ for $i = 1, \dots, n$. Let $q(x) = q(x_1), \dots, q(x_n)$, and let $A = (a_{i,j})$ denote the matrix of the scalars. By Lemma (2.5) we obtain

$$\delta(q(x), q(x)) = \det(A^2)\delta(e, e) \quad \text{in} \quad \Gamma_U^n M.$$

The element $\delta(x, x) \otimes 1$ in $\Gamma_A^n(F) \otimes_{\Gamma_A^n F} U = \Gamma_U^n(F \otimes_A U)$ is invertible by definition. The natural morphism $F \otimes_A U \longrightarrow M$ induces a morphism $\Gamma_U^n(F \otimes_A U) \longrightarrow \Gamma_U^n M$ sending $\delta(x, x) \otimes 1$ to the invertible element $\delta(q(x), q(x))$. Then also $\det(A)$ must be invertible, and consequently we have that $q(x_1), \dots, q(x_n)$ form a basis of $M_A(x)$. $\square$

Definition 5.5. The functor $\mathscr{H}_F^{et}(x)$ is the covariant functor from the category of A-algebras to sets that map an A-algebra B to the set of ideals in $F \otimes_A B$ such that corresponding quotients Q satisfy the following

1. The elements $q(x_1), \dots, q(x_n)$ in Q form a B-module basis, where $q : F \longrightarrow F \otimes_A B \longrightarrow Q$ is the composite map.
2. The algebra homomorphism $B \longrightarrow Q$ is étale.

Lemma 5.6. *Let B be an A-algebra, and Q a B-valued point of $\mathscr{H}_F^{et}(x)$. Then we have the following commutative diagram of algebras*

$$\begin{array}{ccc} \Gamma_A^n F & \longrightarrow & \Gamma_B^n Q \\ \downarrow{\scriptstyle \text{can}} & & \downarrow{\scriptstyle \sigma_Q} \\ U_A(x) \colon = (\Gamma_A^n F)_{\delta(x,x)} & \longrightarrow & B. \end{array} \tag{5.6.1}$$

Proof. The composite morphism $F \longrightarrow F \otimes_A B \longrightarrow Q$ induces a morphism of A-algebras $\Gamma_A^n F \longrightarrow \Gamma_B^n Q$ that sends the element $\delta(x, x)$ to $\delta(q(x), q(x))$, where $q(x) = q(x_1), \dots, q(x_n)$ in Q. By assumption the elements $q(x)$ form a basis of Q and that Q is étale. Then, by Proposition (2.10) we that the image of $\delta(q(x), q(x))$ by the canonical map $\sigma_Q \colon \Gamma_B^n Q \longrightarrow Q$ is a unit, and the commutativity of the diagram (5.6.1) follows. $\square$

5.7. **Universal coefficients.** For each pair of indices $i \leq j$ we look at the product $x_i x_j$ in F, and for each $k = 1, \dots, n$ we consider the sequence

$$x_k^{i,j} = x_1, \dots, x_{k-1}, x_i x_j, x_{k+1}, \dots, x_n \tag{5.7.1}$$

where the k'th element is replaced with the product $x_i x_j$. We now define the universal coefficient

$$\alpha_k^{i,j} = \frac{\delta(x, x_k^{i,j})}{\delta(x, x)} \quad \text{in} \quad U_A(x) = (\Gamma_A^n F)_{\delta(x,x)}. \tag{5.7.2}$$

Proposition 5.8. *Let Q be a B-valued point of $\mathscr{H}_F^{et}(x)$, and let $q\colon F \longrightarrow F \otimes_A B \longrightarrow Q$ denote the composite map. Let $b_k^{i,j}$ be the unique elements in B such that*

$$q(x_i x_j) = \sum_{k=1}^{n} b_k^{i,j} q(x_k)$$

in Q. Then $b_k^{i,j}$ is the specialization of the element $\alpha_k^{i,j}$ under the natural map $U_A(x) \longrightarrow B$ of Lemma (5.6), for each $i, j, k = 1, \dots, n$. In particular we have that $M_A(x) \otimes_{U_A(x)} B = Q$ as quotients of $F \otimes_A B$.

Proof. Having the triplet i, j, k fixed, we let $x_k^{i,j}$ denote the sequence (5.7.1) of elements in F. Consider the element $\delta(q(x), q(x_k^{i,j}))$ in $\Gamma_B^n Q$. We replace the element $q(x_i x_j)$ in Q with $\sum b_k^{i,j} q(x_k)$, and obtain

$$\delta(q(x), q(x_k^{i,j})) = b_k^{i,j} \delta(q(x), q(x)) \quad \in \Gamma_B^n Q.$$

The element $\delta(q(x), q(x))$ is the image of $\delta(x, x)$ by the induced map $\Gamma_A^n F \longrightarrow \Gamma_B^n Q$. It follows from the commutative diagram (5.6.1) that $b_k^{i,j}$ in B is the image of $\alpha_k^{i,j}$. □

Corollary 5.9. *The pair $(U_A(x), M_A(x))$ represents $\mathscr{H}_F^{et}(x)$.*

Proof. It follows from Proposition (5.2) and Lemma (5.4) that $M := M_A(x)$ is a $U := U_A(x)$-valued point of $\mathscr{H}_F^{et}(x)$. If Q is any B-valued point of $\mathscr{H}_F^{et}(x)$ have by Proposition (5.8) one morphism $U \longrightarrow B$ with the desired property, and we need to establish uniqueness of that map. Therefore, let $\varphi_i : U \longrightarrow B$ $(i = 1, 2)$, be two A-algebra homomorphisms such that both extensions $M \otimes_U B$ equal Q as quotients of $F \otimes_A B$. We then have that the natural map

$$\Gamma_U^n M \longrightarrow \Gamma_U^n(M) \otimes_U B = \Gamma_B^n Q$$

is independent of the maps $\varphi_i : U \longrightarrow B$. And in particular the canonical section $\sigma_Q = \sigma_M \otimes 1 : \Gamma_B^n Q \longrightarrow B$ is independent of the maps $\varphi_i, (i = 1, 2)$. For any element $u \in U$ we have that $\sigma_M(u\gamma^n(1)) = u$, and then also that $\sigma_Q(u\gamma^n(1) \otimes 1_B) = \varphi_i(u)$. Thus $\varphi_1 = \varphi_2$, and we have proven uniqueness. □

5.10. **Étale families.** We let $\mathscr{H}_F^{et,n}$ denote the functor of étale families of the Hilbert functor Hilb_F^n of n-points on F. That is, we consider the co-variant functor from A-algebras to sets whose B-valued points are

$$\mathscr{H}_F^{et,n}(B) = \{I \in \mathrm{Hilb}_F^n(B) \mid B \longrightarrow F \otimes_A B/I \text{ is étale}\}.$$

It is clear that $\mathscr{H}_F^{et,n}$ is an open subfunctor of Hilb_F^n and we will end this section by describing the corresponding open subscheme of the Hilbert scheme.

Proposition 5.11. *Let F be an A-algebra. Let $\Delta \subseteq \mathrm{Spec}(\Gamma_A^n F)$ be the closed subscheme defined by the ideal of norms I_F, and and*

let $U = \mathrm{Spec}(\Gamma^n_A F) \setminus \Delta$ *denote its open complement. The family* $\pi_{n|} \colon \pi_n^{-1}(U) \longrightarrow U$ *given in (5.1.1) represents* $\mathscr{H}_F^{et,n}$.

Proof. Clearly the functors $\mathscr{H}_F^{et}(x)$, for different choices of elements $x = x_1, \dots, x_n$ in F, give an open cover of $\mathscr{H}_F^{et,n}$. By Corollary (5.9) the restriction of the family $\pi_{n|} \colon \pi_n^{-1}(U) \longrightarrow U$ to the open subscheme $\mathrm{Spec}(U_A(x)) \subseteq U$ represents $\mathscr{H}_F^{et}(x)$. Thus we need only to check that union of the schemes $\mathrm{Spec}(U_A(x))$, for different $x = x_1, \dots, x_n$, is U, which however is a consequence of Lemma (2.4). □

6. Closure of the locus of distinct points

We will continue with the notation from the preceding sections. In this section we will construct universal families, not for the locus of distinct points as in Section 5, but for its closure.

6.1. **Notation.** Let F be an A-algebra, and let $R = \oplus_{m \geq 0} I_F^m$ denote the graded ring where $I_F \subseteq \Gamma^n_A F$ is the ideal of norms associated to $V = F$. We let $x = x_1, \dots, x_n$ be n-elements in F, and we denote by $R(x) = R_{(\delta(x,x))}$ the degree zero part of the localization of R at $\delta(x,x) \in I_F$. Finally we let $\mathscr{E}$ denote the free $R(x)$-module of rank n. We will write

$$\mathscr{E} = \oplus_{i=1}^n R(x)[x_i], \tag{6.1.1}$$

where $[x_i]$ is our notation for a basis element pointing out the i'th component of the direct sum $\mathscr{E}$. As $\Gamma^n_A F$ is an A-algebra we have that $\mathscr{E}$ is an A-module. We define the A-module homomorphism

$$[\;] \colon F \longrightarrow \mathscr{E} \tag{6.1.2}$$

in the following way. For any $y \in F$, and any $i = 1, \dots, n$, we let

$$x_y^i = x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n \tag{6.1.3}$$

denote the n-elements in F where the i'th element x_i is replaced with y. Then we define the value of the map (6.1.3) on the element $y \in F$ as

$$[y] = \sum_{i=1}^n \frac{\delta(x, x_y^i)}{\delta(x,x)}[x_i] \quad \text{in} \quad \mathscr{E}. \tag{6.1.4}$$

Note that when $y = x_i$ the notation of (6.1.1) is consistent with the notation of (6.1.4). As determinants are linear in its columns (and rows) it follows that the map $[\;] \colon F \longrightarrow \mathscr{E}$ defined above is an A-module homomorphism. Furthermore it follows that we have an induced $R(x)$-module homomorphism

$$[\;] \otimes \mathrm{id} \colon F \otimes_A R(x) \longrightarrow \mathscr{E} \otimes_A R(x) \cong \mathscr{E}, \tag{6.1.5}$$

which is surjective.

6.2. **Universal multiplication.** With the notation as above we define now the $R(x)$-bilinear map $\mathscr{E} \times \mathscr{E} \longrightarrow \mathscr{E}$ by defining its action on the basis as

$$[x_i][x_j] := [x_i x_j] \quad \text{for} \quad i, j \in \{1, \dots, n\}. \tag{6.2.1}$$

We will show that the above defined bilinear map defines as multiplication structure on $\mathscr{E}$ - that is giving $\mathscr{E}$ a structure of a commutative $R(x)$-algebra. We first observe the following simple but important fact. Consider $\mathscr{E}$ as a sheaf on $\mathrm{Spec}(R(x))$, and let $U \subset \mathrm{Spec}(R(x))$ be a subscheme of $\mathrm{Spec}(R(x))$. Assume furthermore that the bilinear map (6.2.1) restricted to $\mathscr{E}_U$ gives a ring structure on $\mathscr{E}_U$. That is the product (6.2.1) is associative, has an multiplicative identity and is distributive, then we also have a ring structure on $\mathscr{E}_{\bar{U}}$, where $\bar{U}$ is the scheme theoretic closure of $U \subseteq \mathrm{Spec}(R(x))$. We will apply this observation to a scheme theoretic dense open subset $U \subseteq \mathrm{Spec}(R(x))$.

Proposition 6.3. *Let F be an A-algebra. We have that (6.1.4) defines an algebra structure on $\mathscr{E}$ and that the map (6.1.5) is a surjective $R(x)$-algebra homomorphism.*

Proof. Let $R = \oplus_{n \geq 0} I_F^n$, where $I_F \subseteq \Gamma_A^n F$ is the ideal of norms. We have that $\mathrm{Spec}(R(x))$ is an affine open subset of $\mathrm{Proj}(R)$, where

$$\rho\colon\ \mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(\Gamma_A^n F)$$

is the blow-up with center $\Delta = \mathrm{Spec}(\Gamma_A^n(F/I_F))$. The open complement $\mathrm{Proj}(R) \setminus \rho^{-1}(\Delta)$ of the effective Cartier divisor $\rho^{-1}(\Delta)$ is schematically dense. Hence

$$U := \mathrm{Spec}(R(x)) \setminus \rho^{-1}(\Delta) \cap \mathrm{Spec}(R(x))$$

is schematically dense in $\mathrm{Spec}(R(x))$. By (6.2) it suffices to show the statements over U. However we have that $U = \mathrm{Spec}(U_A(x))$ as defined in (5.6.1), and that the restriction of $\mathscr{E}_{|U}$ coincides with the family $\mathrm{Spec}(M_A(x))$. In other words, we have that restriction of the multiplication map (6.1.5) to the open U coincides with the universal multiplication map of Proposition (5.11).

□

Corollary 6.4. *We have that $\mathscr{E}(x)$ is an $R(x)$-valued point of the Hilbert functor* Hilb_F^n.

Proof. The proposition gives that $\mathrm{Spec}(\mathscr{E})$ is a closed subscheme of $\mathrm{Spec}(F \otimes_A R(x))$. By construction the $R(x)$-module $\mathscr{E}$ is free of rank n. □

Corollary 6.5. *The schemes* $\mathrm{Spec}(R(x))$*, for different choices of $x = x_1, \dots, x_n$ in F, form an affine open cover of* $\mathrm{Proj}(R)$*, and the families* $\mathrm{Spec}(\mathscr{E}(x)) \longrightarrow \mathrm{Spec}(R(x))$ *glue together to a* $\mathrm{Proj}(R)$*-valued point of the Hilbert functor* Hilb_F^n.

Proof. The first statement follows from Lemma (2.4). To prove the second assertion it suffices to see that the families glue over a open dense set. Let $U = \mathrm{Proj}(R)\backslash\rho^{-1}(\Delta)$, where $\rho\colon \mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ is the blow-up with center Δ. Then we have that $\mathrm{Spec}(R(x)) \cap U = \mathrm{Spec}(U_A(x))$ for any n-elements $x = x_1, \ldots, x_n$ in F, and the result follows. $\square$

7. The good component

7.1. When $X \longrightarrow S$ is an algebraic space we have the Hilbert functor $\mathrm{Hilb}^n_{X/S}$ of closed subspaces of X that are flat and finite of rank n over the base. If $U \longrightarrow X$ is an étale map we define the subfunctor $\mathscr{H}^n_{U\to X}$ of $\mathrm{Hilb}^n_{U/S}$ by assigning to any S-scheme T the set

$$\mathscr{H}^n_{U\to X}(T) = \{Z \in \mathrm{Hilb}^n_{U/S}(T) \text{ such that the composite map } Z \subseteq U \times_S T \longrightarrow X \times_S T \text{ is a closed immersion}\}.$$

Proposition 7.2. *Let $X \longrightarrow S$ be a separated quasi-compact algebraic space over an affine scheme S, and let $U \longrightarrow X$ be an étale representable cover with U an affine scheme, and let $R = U \times_X U$. Then we have the following*

(1) *The functor $\mathscr{H}^n_{U\to X}$ is representable by a scheme.*
(2) *The natural map $\mathscr{H}^n_{U\to X} \longrightarrow \mathrm{Hilb}^n_{X/S}$ is representable, étale and surjective.*
(3) *The two maps $\mathscr{H}^n_{R\to X} \rightrightarrows \mathscr{H}^n_{U\to X}$ form an étale equivalence relation, and the quotient is $\mathrm{Hilb}^n_{X/S}$.*

Proof. Since $X \longrightarrow S$ is separated the composition $Z \longrightarrow U \times_S T \longrightarrow X \times_S T$ will be finite, for any $Z \in \mathrm{Hilb}^n_{U/S}(T)$, any S-scheme T. It is then clear that $\mathscr{H}^n_{U\to X}$ is an open subfunctor of $\mathrm{Hilb}^n_{U/S}$ where the latter is known to be representable ([12]). This shows the first assertion. To see that the map $\mathscr{H}^n_{U\to X} \longrightarrow \mathrm{Hilb}^n_{X/S}$ is representable we let $T \longrightarrow \mathrm{Hilb}^n_{X/S}$ be a morphism, with T some S-scheme. Let $Z \subseteq X \times_S T$ denote the corresponding closed subscheme, and let $Z_U = Z \times_X U$. It is easily verified that the fiber product $\mathscr{H}^n_{U\to X} \times_{\mathrm{Hilb}^n_{X/S}} T$ equals the set of sections of $Z_U \longrightarrow Z$. Thus the fibred product equals the Weil restriction of scalars $\mathfrak{R}_{Z/T}(Z_U)$ of Z_U with respect to $Z \longrightarrow T$. The fiber of $Z_U \longrightarrow T$ over any point in T, is an affine scheme, and it follows from [4, Thm.4] that the Weil restriction $\mathfrak{R}_{Z/T}(Z_U)$ is representable by a scheme. Hence the map $\mathscr{H}^n_{U\to X} \longrightarrow \mathrm{Hilb}^n_{X/S}$ is representable. Étaleness of the map follows from [4, Prop. 5], and surjectivity follows as any T-valued point of $\mathrm{Hilb}^n_{X/S}$ étale locally lifts to U. The last assertion follows as it is easy to see that the natural map $\mathscr{H}^n_{R\to X} \longrightarrow \mathscr{H}^n_{U\to X} \times_{\mathrm{Hilb}^n_{X/S}} \mathscr{H}^n_{U\to X}$ is in fact an isomorphism. $\square$

Corollary 7.3. *Let $X \longrightarrow S$ be a separated map of algebraic spaces. Then $\mathrm{Hilb}^n_{X/S}$ is an algebraic space.*

Proof. It suffices to show the statement for affine base S. Let $X' \subseteq X$ be an open immersion. Then as $X \longrightarrow S$ is assumed separated we have a map $\mathrm{Hilb}^n_{X'/S} \longrightarrow \mathrm{Hilb}^n_{X/S}$ which is a representable open immersion. Furthermore, as

$$\mathrm{Hilb}^n_{X/S} = \lim_{\substack{X' \subseteq X \\ \text{open, q-compact}}} \mathrm{Hilb}^n_{X'/S}$$

we may assume that $X \longrightarrow S$ is quasi-compact as well. Then the result follows from the proposition. □

Remark 7.4. For a quasi-projective scheme $X \longrightarrow S$ over a Noetherian base scheme S it was proven by Grothendieck that the Hilbert functor $\mathrm{Hilb}^n_{X/S}$ is representable by a scheme ([11]). For a separated algebraic space $X \longrightarrow S$ locally of finite type, Artin proved that $\mathrm{Hilb}^n_{X/S}$ is an algebraic space ([2]). The proof of the general result above showing that $\mathrm{Hilb}^n_{X/S}$ is an algebraic space for any separated algebraic space $X \longrightarrow S$ was suggested to us by the referee.

7.5. **The good component.** Let $X \longrightarrow S$ be a separated algebraic space, and let $Z \longrightarrow \mathrm{Hilb}^n_{X/S}$ be the universal family, which by definition is finite, flat of rank n. The discriminant $D_Z \subseteq \mathrm{Hilb}^n_{X/S}$ is a closed subspace with the open complement $U^{et}_{X/S}$ parameterizing length n étale subspaces of X. We define $\mathrm{G}^n_{X/S} \subseteq \mathrm{Hilb}^n_{X/S}$ as the schematic closure of the open subspace $U^{et}_{X/S}$. We call $\mathrm{G}^n_{X/S}$ the *good* or *principal component.*

Remark 7.6. Let $f\colon Z \longrightarrow H$ be a morphism of algebraic spaces which is a finite and flat morphism of rank n. Then the set $U \subseteq H$ where f is étale is an open subset being the complement of the discriminant $D_{Z/H}$. The scheme theoretic closure of $U \subseteq H$ is then the largest closed subscheme of H over which the discriminant of f is a non-zero-divisor.

Theorem 7.7. *Let $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ be a morphism of affine schemes, and let $\Delta \subseteq \mathrm{Spec}(\Gamma^n_A F)$ be the closed subscheme defined by the ideal of norms. Then we have that the good component $\mathrm{G}^n_{X/S}$ is isomorphic to the blow-up $\mathrm{Bl}(\Delta)$ of $\mathrm{Spec}(\Gamma^n_A F)$ along Δ. The isomorphism*

$$\mathrm{b}_X\colon \mathrm{G}^n_{X/S} \xrightarrow{\simeq} \mathrm{Bl}(\Delta),$$

is induced from restricting the norm map $\mathrm{n}_X\colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ to the good component $\mathrm{G}^n_{X/S}$.

Proof. By Theorem (4.10) we have that the inverse image $\mathrm{n}_X^{-1}(\Delta)$ is the discriminant $D_Z \subseteq \mathrm{Hilb}^n_{X/S}$ of the universal family $Z \longrightarrow \mathrm{Hilb}^n_X$. Consequently we have that the local equation of the closed immersion

$$\mathrm{G}^n_{X/S} \cap \mathrm{n}_X^{-1}(\Delta) \subseteq \mathrm{G}^n_{X/S},$$

is not a zero divisor. Therefore, by the universal properties of the blow-up, we get an induced morphism $\mathrm{b}_X\colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}(\Delta)$. A morphism we will show is an isomorphism.

We have by Corollary (6.5) the $\mathrm{Bl}(\Delta)$-valued point $\mathscr{E}$ of the Hilbert functor Hilb^n_F. From the defining properties of the Hilbert scheme we then have a morphism $f_{\mathscr{E}}\colon \mathrm{Bl}(\Delta) \longrightarrow \mathrm{Hilb}^n_{X/S}$ such that the pull-back of the universal family is $\mathscr{E}$. When restricting $\mathscr{E}$ to the open set $U = \mathrm{Spec}(\Gamma^n_A F) \setminus \Delta$ we have an étale family – by construction of $\mathscr{E}$. Hence the image $f_{\mathscr{E}}(U)$ is contained in $U^{et}_{X/S}$. It follows that the schematically closure $\overline{U^{et}_{X/S}} = \mathrm{G}^n_{X/S}$ contains the image of the closure of $\overline{U} = \mathrm{Bl}(\Delta)$. Consequently we have a morphism $f_{\mathscr{E}}\colon \mathrm{Bl}(\Delta) \longrightarrow \mathrm{G}^n_{X/S}$, a morphism we claim is the inverse to the map $\mathrm{b}_X\colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}(\Delta)$.

By Proposition (5.11) we have that the restriction of $f_{\mathscr{E}}$ to U is the inverse of the restriction of b_X to $U^{et}_{X/S}$. As both U in $\mathrm{Bl}(\Delta)$ and $U^{et}_{X/S}$ in $\mathrm{G}^n_{X/S}$ are open complements of effective Cartier divisors it follows that $f_{\mathscr{E}}$ is the inverse of b_X. □

7.8. For a separated map of algebraic spaces $X \longrightarrow S$ there exists an algebraic space $\Gamma^n_{X/S}$ that naturally globalize the affine situation with $\mathrm{Spec}(\Gamma^n_A F)$ ([20]). For the convenience of the reader we will give a description of this space for X quasi-compact over an affine base. Not only is the quasi-compact case technically easier to handle, but it turns out to be sufficient in order to generalize Theorem (7.7) for separated algebraic spaces $X \longrightarrow S$.

7.9. **Pro-equivalence.** We will say that two sequences (indexed by the non-negative integers) of ideals $\{I_m\}$ and $\{J_m\}$ in a ring B are *pro-equivalent* if there exists an integer p such $I_{m+p} \subseteq J_m$, and $J_{m+p} \subseteq I_m$, for all $m \geq 0$.

Lemma 7.10. *Let G be a finite group acting on a Noetherian ring B and let $\mathfrak{a} \subseteq B$ be an invariant ideal. Assume furthermore that the invariant ring B^G is Noetherian, and that B is a finite module over the invariant ring. Then $\{(\mathfrak{a}^G)^m\}$ is pro-equivalent with $\{(\mathfrak{a}^m)^G\}$.*

Proof. Clearly $(\mathfrak{a}^G)^{m+p} \subseteq (\mathfrak{a}^m)^G$, and consequently it suffices to show that $(\mathfrak{a}^{m+p})^G \subseteq (\mathfrak{a}^G)^m$ for some p. An element $x \in B$ is a root of the monic polynomial $m_x(t) = \prod_{g \in G}(t - g.x)$. Since $\mathfrak{a}$ is G-invariant we have for any $x \in \mathfrak{a}$ that $x^{|G|} \in \mathfrak{a}^G$, and consequently that

$$\mathfrak{a}^{m+|G|} \subseteq (\mathfrak{a}^G)^m B.$$

By assumption B is a finitely generated B^G-module, and consequently by the Artin-Rees Lemma ([3, Cor. 10.10]) there exists an integer k such that

$$(\mathfrak{a}^G)^m B \cap B^G = (\mathfrak{a}^G)^{m-k}\big((\mathfrak{a}^G)^k B \cap B^G\big) \subseteq (\mathfrak{a}^G)^{m-k}.$$

Hence $(\mathfrak{a}^{m+|G|+k})^G \subseteq (\mathfrak{a}^G)^m$ for all $m \geq 0$. $\square$

Lemma 7.11. *Let F be an A-algebra of finite type, and let $I \subseteq F$ be an ideal of finite type. For each $m > 0$ we let J_m denote the kernel of the natural map $\Gamma^n_A(F) \longrightarrow \Gamma^n_A(F/I^m)$. Then $\{J_m\}$ is pro-equivalent with $\{J_1^m\}$.*

Proof. We first show the case with a polynomial ring over the integers. Let $X = x_1, \dots, x_r$ and $T = t_1, \dots, t_s$ be variables over $\mathbf{Z}$, and let $F = \mathbf{Z}[X, T]$, and $I = (T)$. Let $\mathfrak{a}_m$ denote the kernel of $\mathrm{T}^n_A F \longrightarrow \mathrm{T}^n_A(F/I^m)$. It is easily checked that $\{\mathfrak{a}_m\}$ is pro-equivalent with $\{\mathfrak{a}_1^m\}$. The group $\mathfrak{S}_n$ acts on $\mathrm{T}^n_A F$, and it follows that $\{(\mathfrak{a}_1^m)^{\mathfrak{S}_n}\}$ is pro-equivalent with $\{\mathfrak{a}_m^{\mathfrak{S}_n}\}$. By Lemma (7.10) we have that $\{(\mathfrak{a}_1^m)^{\mathfrak{S}_n}\}$ is pro-equivalent with $\{(\mathfrak{a}_1^{\mathfrak{S}_n})^m\}$. As F/I^m is free, an in particular flat $\mathbf{Z}$-module for all $m > 0$, we have that $\Gamma^n_A(F/I^m) = \mathrm{TS}^n_A(F/I^m)$. In particular we get that

$$\ker(\Gamma^n_A F \longrightarrow \Gamma^n_A(F/I^m)) = (\mathfrak{a}_m)^{\mathfrak{S}_n},$$

and we have proven the lemma in the special case. Since $\Gamma^n_{\mathbf{Z}}\mathbf{Z}[X,T] \otimes_{\mathbf{Z}} A = \Gamma^n_A A[X,T]$ we have also proven the lemma for $F = A[X,T]$, and $I = (T)$. In the general case we let $\varphi \colon A[X,T] \longrightarrow F$ denote the A-algebra homomorphism that sends X to a set of generators of F, and T to a set of generators of the ideal $I \subseteq F$. For each $m > 0$ we have induced surjective maps $\varphi_m \colon A[X,T]/(T)^m \longrightarrow F/I^m$ and $\Gamma(\varphi_m) \colon \Gamma^n_A A[X,T]/(T)^m \longrightarrow \Gamma^n_A F/I^m$. An element in $\ker(\Gamma(\varphi_m))$ is of the form ([18, Prop. IV.8, p. 284])

$$\gamma^c(\bar{f}) * \gamma^{n-c}(\bar{g})$$

where $\bar{g} \in A[X,T]/(T)^m$ and $\bar{f} \in \ker(\varphi_m)$. Clearly we can find elements f and g in $A[X,T]$, with $f \in \ker(\varphi)$, that restricts to $\bar{f}$ and $\bar{g}$ by the canonical map. Thus the induced map $\ker(\Gamma^n(\varphi)) \longrightarrow \ker(\Gamma^n(\varphi_m))$ is surjective for all $m > 0$. It follows that the induced map from

$$\mathfrak{a}_m = \ker\left(\Gamma^n_A A[X,T] \longrightarrow \Gamma^n_A(A[X,T]/(T)^m)\right)$$

to $J_m = \ker(\Gamma^n_A F \longrightarrow \Gamma^n_A(F/I^m))$ is surjective. In particular $\mathfrak{a}_1$ surjects to J_1, so $\mathfrak{a}_1^m$ surjects to J_1^m. The lemma now follows by lifting elements to $\mathfrak{a}_m$ and $\mathfrak{a}_1^m$, where the result holds. $\square$

7.12. **FPR-sets.** Let G be a finite group acting on a separated algebraic space X. By a result of Deligne the geometric quotient X/G exists as an algebraic space. We will make use of that result, but we need also to recall some terminology: For any group element $\sigma \in G$ we have the induced map $(\mathrm{id}_X, \sigma) \colon X \longrightarrow X \times_S X$. By taking the intersection of the diagonal and X via the map (id_X, σ) we get a closed subspace $X^\sigma \subseteq X$. If $f \colon X \longrightarrow Y$ is a G equivariant map, we have a closed immersion $X^\sigma \subseteq f^{-1}(Y^\sigma)$. An equivariant map $f \colon X \longrightarrow Y$ is *fixed-point-reflecting* (abbreviated FPR) if we have an equality of *sets*

$X^\sigma = f^{-1}(Y^\sigma)$ for all $\sigma \in G$ ([15, p.183]). We have an alternative description of this condition: For every point $x \in X$ the stabiliser group G_x is equal to the stabiliser group G_y of the image point $y = f(x)$ (in general we only have the inclusion $G_x \subseteq G_y$). This allows us to say that f is *fixed-point-reflecting at x* (abbreviated FPR at x) if $G_x = G_y$. We then have some general facts about FPR-sets.

Lemma 7.13. *i) Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are G-morphisms, h their composite and $x \in X$. Then h is FPR at x precisely when f is FPR at x and g is FPR at $f(x)$.*

ii) Suppose that $\{X_\alpha\}$ is an inverse system with affine transition maps of G-spaces. For $x \in X := \varprojlim_\alpha X_\alpha$ the set $S_x := \{\alpha \mid p_\alpha$ is FPR at $x\}$, where $p_\alpha\colon X \to X_\alpha$ is the structure map, is non-empty and upwards closed *(i.e., if $\alpha \in S_x$ and $\alpha' \geq \alpha$ then $\alpha' \in S_x$).*

iii) Suppose now that also $\{Y_\alpha\}$ is an inverse system with affine transition maps of G-spaces over the same index set and that $\{f_\alpha\colon X_\alpha \to Y_\alpha\}$ is a G-morphism of directed systems. Set $Y := \varprojlim_\alpha Y_\alpha$, $f := \varprojlim_\alpha f_\alpha$ and assume that f is FPR at $x \in X$. Then $\{\alpha \mid f_\alpha$ is FPR at $x_\alpha\}$ is non-empty and upwards closed.

Proof. For the first part we always have that $G_x \subseteq G_{f(x)} \subseteq G_{h(x)}$ so that if h is FPR at x, i.e., $G_x = G_{h(x)}$, then f is FPR at x and g is FPR at $f(x)$ and clearly conversely.

That S_x is upwards directed follows from i). For every $g \notin G_x$ we have $gx \neq x$ and hence there is an index α_g such that $gx_{\alpha_g} \neq x_{\alpha_g}$, where for all indices β, $x_\beta := p_\beta(x)$. Picking an α such that $\alpha \geq \alpha_g$ for all such g we get that $gx_\alpha \neq x_\alpha$ for all $g \in G_x$ which implies that $G_{x_\alpha} = G_x$ so that $\alpha \in S_x$.

Finally, iii) follows from i) and ii) applied to $f(x) \in Y$. $\square$

Definition-Lemma 7.14. *If the equivariant map $f\colon X \longrightarrow Y$ is separated and unramified, then X^σ is both open and closed in $f^{-1}(Y^\sigma)$. Hence if Y is also separated over some S on which G acts trivially, there is a maximal open FPR-subset of X, which we call the* FPR-locus *of f.*

In the particular case when $U \longrightarrow X$ is an unramified separated map and X is separated over S, we will denote the FPR-locus of the induced $\mathfrak{S}_n$-map $U^n_S \longrightarrow X^n_S$ by $\Omega_{U\to X} \subseteq U^n_S$.

Proof. We have a map $f^{-1}(Y^\sigma) \to X \times_Y X$ given by $x \mapsto (x, \sigma x)$ and X^σ is the inverse image of the diagonal. As f is unramified and separated, the diagonal is open and closed in $X \times_Y X$ and hence so is X^σ in $f^{-1}(Y^\sigma)$. If Y is also separated, then $f^{-1}(Y^\sigma)$ is closed in X and hence the complement of X^σ in $f^{-1}(Y^\sigma)$ is closed in X and removing such subsets for all σ gives the FPR-locus. $\square$

Lemma 7.15. *Let $F \longrightarrow F'$ be an étale extension of A-algebras. Let $\varphi\colon \mathrm{T}^n_A F' \longrightarrow L$ be a map to a field L and let $\varphi_i\colon F' \longrightarrow L$ be the co-projections of φ (with $i = 1, \ldots, n$). Define the ideals $J = \cap \ker \varphi_i$ in F' and $I = \cap \ker \varphi_{i|F}$ in F. If the point φ is in the fixed point reflecting set $\Omega_{F\to F'}$ of $\mathrm{Spec}(\mathrm{T}^n_A F') \longrightarrow \mathrm{Spec}(\mathrm{T}^n_A F)$, then the induced map*

$$F/I^m \longrightarrow F'/J^m$$

is an isomorphism, for all $m > 0$.

Proof. We may work étale locally around the points given by $\ker \varphi_i$ and $\ker \varphi_{i|F}$ so we may assume that F is a product of strictly Henselian rings with $\ker \varphi_{i|F}$ as maximal ideals and similarly for F'. That $\varphi \in \Omega_{F\to F'}$ means that the map $F \to F'$ induces a bijection on maximal ideals, which as F is semi-local strictly Henselian and $F \to F'$ is étale means that $F \to F'$ is an isomorphism. From this the lemma follows immediately. ☐

7.16. **Notation.** When $U \longrightarrow X$ is an étale cover, we let

$$\Omega'_{U\to X} \subseteq U^n_S/\mathfrak{S}_n$$

be the image of the FPR-locus $\Omega_{U\to X} \subseteq U^n_S$ by the quotient map. Set $R = U \times_X U$, then the FPR-locus $\Omega_{R\to X}$ is identified with $\Omega_{U\to X} \times_{X^n_S} \Omega_{U\to X}$, so in particular we have that ([15, p. 183-184])

$$\Omega'_{R\to X} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \Omega'_{U\to X} \tag{7.16.1}$$

is an étale equivalence relation with quotient $X^n_S/\mathfrak{S}_n$.

Assume now that the base $S = \mathrm{Spec}(A)$ is affine, and that X is a quasi-compact algebraic space. Let $U = \mathrm{Spec}(F) \longrightarrow X$ be an étale affine cover. The map $\mathrm{Spec}(\alpha_n)\colon \mathrm{Spec}(\mathrm{TS}^n_A F) \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ is a (universal) homeomorphism (see e.g. [20, Corollary 4.2.5]), and we let

$$\Omega''_{U\to X} \subseteq \mathrm{Spec}(\Gamma^n_A F)$$

denote the open set given as the image of $\Omega'_{U\to X}$ by $\mathrm{Spec}(\alpha_n)$. Our first aim is to prove that the homeomorphic image of (7.16.1) also forms an étale equivalence relation.

Proposition 7.17. *Let $F \longrightarrow F'$ be an étale extension of A-algebras. Let $\xi \in \Omega''_{F\to F'}$ be a point of A. Then the induced map of completions*

$$(\Gamma^n_A F)_{\widehat{f(\xi)}} \longrightarrow (\Gamma^n_A F')_{\widehat{\xi}}$$

is an isomorphism, where $f(\xi)$ is the image of ξ by the induced map $\mathrm{Spec}(\Gamma^n_A F') \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$.

Proof. It suffices to show that there are ideals $I_1 \subset \Gamma^n_A F$ and $J_1 \subset \Gamma^n_A F'$ contained in the ideals corresponding to the points $f(\xi)$ and ξ, respectively, such that the induced map of formal neighborhoods

$$\varprojlim (\Gamma^n_A F)/I_1^m \longrightarrow \varprojlim (\Gamma^n_A F')/J_1^m \tag{7.17.1}$$

is an isomorphism. As the morphism $\mathrm{Spec}(\Gamma^n_A F') \longrightarrow \mathrm{Spec}(\mathrm{TS}^n_A F')$ is a homeomorphism the point ξ lifts to a point of $\mathrm{Spec}(\mathrm{T}^n_A F')$. Let $\varphi\colon \mathrm{T}^n_A F' \longrightarrow L'$ be a lifting of $\xi = \mathrm{Spec}(L)$, with L' some field extension of L. Write $\varphi = (\varphi_1, \dots, \varphi_n)$, and define the ideal $J = \cap \ker(\varphi_i)$ in F. We let $J_m = \ker(\Gamma^n_A F' \longrightarrow \Gamma^n_A(F'/J^m)$. As the map $\Gamma^n_A F' \longrightarrow L$ factors via $\Gamma^n_A(F/J)$ we have that J_1 is contained in the ideal $\ker(\Gamma^n_A F' \longrightarrow L)$. We let $I_m = \ker(\Gamma^n_A F \longrightarrow \Gamma^n_A(F/I^m))$ where $I = \cap \ker(\varphi_{i|F})$, and we consider the induced map (7.17.1).

By Lemma (7.11) we have the limit of the system $\{(\Gamma^n_A F)/I_1^m\}$ equals the limit of the system $\{(\Gamma^n_A F)/I_m = \Gamma^n_A(F/I^m)\}$. By Lemma (7.15) we have that $F/I^m = F'/J^m$, and it follows that the map (7.17.1) is an isomorphism. □

Corollary 7.18. *Let $F \longrightarrow F'$ be an étale extension of A-algebras, and let $I_F \subseteq \Gamma^n_A F$ and $I_{F'} \subseteq \Gamma^n_A F'$ be the ideal of norms associated to F and F', respectively. These two ideals, $I_F \Gamma^n_A F'$ and $I_{F'}$, are equal when restricted to the FPR-set $\Omega''_{F \to F'} \subseteq \mathrm{Spec}(\Gamma^n_A F')$.*

Proof. Assume first that the result is true when F (and hence F') is a finitely presented A-algebra. We can write $f\colon F \longrightarrow F'$ as a limit by a directed set of étale maps $f_\alpha\colon F_\alpha \longrightarrow F'_\alpha$ of finitely presented A-algebras, such that $F'_\alpha \otimes_{F_\alpha} F_\beta \backsimeq F'_\beta$ for all α and all $\beta \geq \alpha$. This means that $\mathrm{Spec}(\mathrm{T}^n_A F') \to \mathrm{Spec}(\mathrm{T}^n_A F)$ can be thought of as $\varprojlim_\beta \mathrm{Spec}(\mathrm{T}^n_A F'_\beta) \to \varprojlim_\beta \mathrm{Spec}(\mathrm{T}^n_A F_\beta)$ and similarly for T^n replaced by TS^n (as directed direct limits commute with taking invariants) and Γ^n. The equality to be proven is one of equality of stalks so we may focus on a particular point $x'' \in \Omega''_{F \to F'}$ which is the image of some $x \in \Omega_{F \to F'}$. By Lemma (7.13) we may assume that all projection maps $\mathrm{Spec}(\mathrm{T}^n_A F) \to \mathrm{Spec}(\mathrm{T}^n_A F_\alpha)$ are FPR at the image of x in $\mathrm{Spec}(\mathrm{T}^n_A F)$ and hence, again by Lemma (7.13), we get that $\mathrm{Spec}(\mathrm{T}^n_A F') \to \mathrm{Spec}(\mathrm{T}^n_A F)$ is FPR at $p_\alpha(x)$ for all α which means that $p_\alpha(x'') \in \Omega''_{F_\alpha \to F'_\alpha}$. Hence we get that the equality $I_{F_\alpha} \Gamma^n_A F' = I_{F'_\alpha} \Gamma^n_A F'$ at x'' and taking the direct limit of sheaves in α gives the Corollary at x'' and hence in $\Omega''_{F \to F'}$.

We are therefore left with the case when F is a finitely presented A-algebra. By another (simpler) limit argument we reduce to the case when A is Noetherian. To show equality of the two ideals I'_F and $I_F \Gamma^n_A F'$ it now suffices to show equality in the completion $(\Gamma^n_A F')_{\widehat{\xi}}$, for each closed point $\xi \in \Omega_{F \to F''}$ lying over the maximal ideal of A. The result now follows from the proposition. □

Corollary 7.19. *Let $F \longrightarrow F'$ be an étale extension of A-algebras. The induced map $\Omega''_{F \to F'} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ is étale.*

Proof. By doing the same reductions as in the previous corollary, we may assume that A is Noetherian and F and F' are finite type A-algebras. By localisation and Henselisation in A we may then also

assume that A is strictly Henselian. The result then follows from the proposition. $\square$

Lemma 7.20. *Let $X \longrightarrow S$ be a quasi-compact separated algebraic space over an affine base S. Write X as a quotient $R \rightrightarrows U$, with affine schemes U and R. Then we have that $\Omega''_{R\to X} \rightrightarrows \Omega''_{U\to X}$ is an étale equivalence relation.*

Proof. We have (7.16.1) that $\Omega'_{R\to X} \rightrightarrows \Omega'_{U\to X}$ is an étale equivalence relation. As the map $\Omega'_{U\to X} \longrightarrow \Omega''_{U\to X}$ is a homeomorphism, we have that the induced map $\Omega''_{R\to X} \longrightarrow \Omega''_{U\to X} \times_S \Omega''_{U\to X}$ is injective over $\mathrm{Spec}(k)$-valued points, with k a field. To prove the lemma it then suffices to show that the two maps $\Omega''_{R\to X} \rightrightarrows \Omega''_{U\to X}$ are étale. The projection maps $p_i \colon R \longrightarrow U$ are étale, and we have open immersions $\Omega_{R\to X} \subseteq \Omega_{p_i \colon R\to U}$. Étaleness of the two maps $\Omega''_{R\to X} \rightrightarrows \Omega''_{U\to X}$ then follows from Corollary (7.19). $\square$

Proposition 7.21. *Let $X \longrightarrow S$ be a separated quasi-compact algebraic space over an affine scheme $S = \mathrm{Spec}(A)$. Let $U = \mathrm{Spec}(F) \longrightarrow X$ be an étale affine cover, and let $R = U \times_X U$. Define $\Gamma^n_{X/S}$ as the quotient of the étale equivalence relation $\Omega''_{R\to X} \rightrightarrows \Omega''_{U\to X}$.*

(1) *We have a cartesian diagram*

$$\begin{tikzcd}
\mathscr{H}^n_{U\to X} \arrow[r] \arrow[d, "\mathrm{n}_U"] & \mathrm{Hilb}^n_{U/S} \arrow[d, "\mathrm{n}_U"] \\
\Omega''_{U\to X} \arrow[r] & \Gamma^n_{U/S} = \mathrm{Spec}(\Gamma^n_A(F)).
\end{tikzcd}$$

(2) *In the diagram below we have $n_U \circ p_i = q_i \circ n_R$, $i = 1, 2$, and consequently there is an induced map $\mathrm{n}_X \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \Gamma^n_{X/S}$:*

$$\begin{tikzcd}
\mathscr{H}^n_{R\to X} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\mathrm{n}_R"] & \mathscr{H}^n_{U\to X} \arrow[r, "p"] \arrow[d, "\mathrm{n}_U"] & \mathrm{Hilb}^n_{X/S} \arrow[d, "\mathrm{n}_X"] \\
\Omega''_{R\to X} \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & \Omega''_{U\to X} \arrow[r, "q"] & \Gamma^n_{X/S}
\end{tikzcd}$$

Moreover, the commutative diagrams above are cartesian.

Proof. Let us first consider the special case with $S = \mathrm{Spec}(k)$, where k is an algebraically closed field. A k-valued point $Z \subseteq U$ of the Hilbert functor $\mathrm{Hilb}^n_{U/S}$ has support at a finite number of points $\xi_1, \dots, \xi_p$. By (4.4) the associated cycle $\mathrm{n}_U(Z)$ consist of the points $\xi_1, \dots, \xi_p$ counted with multiplicities $m_1, \dots, m_p$. We have that the cycle $n_U(Z)$ is in the FPR-set $\Omega''_{U\to X}$ if and only if the closed subscheme $Z \subseteq U$ also is a closed subscheme of X.

Now, let us prove the proposition. In the first diagram (1) the horizontal maps are open immersions. To see that it is commutative and

cartesian it suffices to establish the equality of the two open sets $\mathscr{H}^n_{U\to X}$ and $\mathrm{n}_U^{-1}(\Omega''_{U\to X})$ of $\mathrm{Hilb}^n_{U/S}$. This we can be checked by reducing to $S=\mathrm{Spec}(k)$, with k algebraically closed. Then we are in the special case considered above from which Assertion (1) follows.

In particular we have proven that the restriction of the norm map n_U to the open subset $\mathscr{H}^n_{U\to X}$ has $\Omega''_{U\to X}$ as domain. We therefore obtain the two leftmost diagrams in (2). Since the horizontal maps in these diagram are étale (Proposition (7.2) and Lemma (7.20)) we can prove the diagrams are cartesian by evaluation over algebraically closed points. We are then again reduced to the special case considered above, which proves assertions in (2). □

Proposition 7.22 (Rydh)**.** *Let $X \longrightarrow S$ be a separated map of algebraic spaces. Then there exists an algebraic space $\Gamma^n_{X/S} \longrightarrow S$ such that*

1. *When $X \longrightarrow S$ is quasi-compact with S an affine scheme, the space $\Gamma^n_{X/S}$ coincides with the one constructed above (7.21).*
2. *For any base change map $T \longrightarrow S$ we have a natural identification $\Gamma^n_{X/S}\times_S T = \Gamma^n_{X\times_S T/T}$.*
3. *For any open immersion $X' \subseteq X$ we have an open immersion $\Gamma^n_{X'/S} \subseteq \Gamma^n_{X/S}$, and moreover*
$$\Gamma^n_{X/S} = \varinjlim_{\substack{X'\subseteq X \\ \text{open, q-compact}}} \Gamma^n_{X'/S}.$$
4. *There is a universal homeomorphism $X^n_S/\mathfrak{S}_n \longrightarrow \Gamma^n_{X/S}$, which is an isomorphism when $X \longrightarrow S$ is flat, or when the characteristic is zero.*

Proof. All results can be found in ([20]): Existence of the space $\Gamma^n_{X/S}$ is Theorem (3.4.1), whereas Assertion (4) is Corollary (4.2.5), and the statement about open immersions in (3) is a special case of Proposition (3.1.7). The functorial description of $\Gamma^n_{X/S}$ given by David Rydh immediately gives assertion (2) and that $\Gamma^n_{X/S}$ is the union of $\Gamma^n_{X'/S}$ with quasi-compact $X' \subseteq X$. Assertion (1) follows as our $\Omega''_{U\to X}$ is what Rydh denotes with $\Gamma^n(U/S)_{|\mathrm{reg}/f}$ (see Proposition (4.2.4), and the proof of Theorem (3.4.1), loc. cit.). □

7.23. **The ideal sheaf of norms.** For $X \longrightarrow S$ quasi-compact and separated over an affine base we have by Corollary (7.18) that the ideals of norms patch together to form an ideal sheaf $\mathscr{I}_X$ on $\Gamma^n_{X/S}$. As these ideals clearly commute with open immersions and base change we obtain by (3) and (1) of Proposition (7.22), an ideal sheaf of norms $\mathscr{I}_X$ on $\Gamma^n_{X/S}$, for any separated algebraic space $X \longrightarrow S$. Let
$$\Delta_X \subseteq \Gamma^n_{X/S}$$
denote the closed subspace defined by the ideal sheaf of norms.

Theorem 7.24. *Let $X \longrightarrow S$ be a separated morphism of algebraic spaces. Then the good component $\mathrm{G}^n_{X/S}$ of $\mathrm{Hilb}^n_{X/S}$ is isomorphic to the blow-up of $\Gamma^n_{X/S}$ along the closed subspace $\Delta_X \subseteq \Gamma^n_{X/S}$, defined by the ideal of norms associated to $X \longrightarrow S$. Moreover, if $X \longrightarrow S$ is flat then $\mathrm{G}^n_{X/S}$ is obtained by blowing-up the geometric quotient $X^n_S/\mathfrak{S}_n$.*

Proof. The Hilbert scheme $\mathrm{Hilb}^n_{X/S}$ and $\Gamma^n_{X/S}$ commute with arbitrary base change. The good component $\mathrm{G}^n_{X/S}$ as well as blow-ups, commute with flat, and in particular étale base change. We may therefore assume that the base S is an affine scheme.

For any open immersion $X' \subseteq X$, with X' quasi-compact, we have a norm map $\mathrm{n}_{X'} \colon \mathrm{Hilb}^n_{X'/S} \longrightarrow \Gamma^n_{X'/S}$ which, by varying X', form a norm map $\mathrm{n}_X \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \Gamma^n_{X/S}$. We claim now that the inverse image $\mathrm{n}_X^{-1}(\Delta_X)$ is locally principal, which we can verify on an open cover. Moreover, given that we obtain an induced map from the good component $\mathrm{G}^n_{X/S}$ to the blow-up of $\Gamma^n_{X/S}$ along Δ_X. To verify that the induced map is an isomorphism, we also reduce to an open cover. Consequently we may assume that X itself is quasi-compact.

When X is quasi-compact we choose an étale affine cover $U \longrightarrow X$. Then by using the cartesian diagrams (2) and (1) of Proposition (7.21) one establish using Theorem (4.10) that $\mathrm{n}_X^{-1}(\Delta_X)$ is locally principal. By Theorem (7.7) we have that the blow-up of $\Delta_U \subseteq \Gamma^n_{U/S}$ yields the good component $\mathrm{G}^n_{U/S}$, and the isomorphism is given induced by the norm map n_U. It then follows by the two cartesian diagrams (2) and (1) of Proposition (7.21), that the map induced map from $\mathrm{G}^n_{X/S}$ to the blow-up of $\Delta_X \subseteq \Gamma^n_{X/S}$ is an isomorphism. □

7.25. **The case of surfaces.** Before we give a corollary to this result we need a generalisation of a result of Fogarty on the smoothness of the Hilbert scheme ([10, Theorem 2.9]). Fogarty proves that the Hilbert scheme of a smooth map $X \longrightarrow S$ is smooth of relative dimension 2 provided that S is a Dedekind scheme. As the Hilbert scheme commutes with base change and flatness can be verified in the integral case by pulling back to Dedekind bases it follows that the result of Fogarty is valid when the base S is integral. However, as we will see, no conditions on the base is needed for that statement. We shall give a direct proof by proving formal smoothness using the infinitesimal lifting criterion and the Hilbert-Burch theorem.

Proposition 7.26. *Let $X \longrightarrow S$ be a smooth and separated morphism of relative dimension* 2*. Then $\mathrm{Hilb}^n_{X/S} \longrightarrow S$ is smooth for all n.*

Proof. As $\mathrm{Hilb}^n_{X/S}$ commutes with base change we can assume that the base is Noetherian. It is enough to show formal smoothness so the statement would follow if we could show that for every small thickening $T \subset T'$ of local Artinian S-schemes, any T-flat finite subscheme $Z \subseteq$

$X \times_S T$ can be extended to a T'-flat finite subscheme of $X \times_S T'$. Let s be the closed point in S. The obstruction for the existence of such a lifting is an element $\alpha \in \mathrm{Ext}^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})$. We have an exact "local-to-global" sequence

$$H^1(X_s, \mathscr{H}om_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})) \to \mathrm{Ext}^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s}) \to H^0(X_s, \mathscr{E}xt^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})).$$

As $\mathscr{H}om_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})$ has finite support, the left term of the above sequence is 0, and consequently it suffices to show that the image of the obstruction element α in $H^0(X_s, \mathscr{E}xt^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s}))$ is zero. As Z is a disjoint union of points we have that $\alpha = \prod \alpha_{z_i}$, where at a point $z \in Z$ the factor α_z is the obstruction for lifting $\operatorname{Spec} \mathscr{O}_{Z,z}$, which is a closed flat subscheme of $\operatorname{Spec} \mathscr{O}_{X \times_S T,z}$, to a flat subscheme of $\operatorname{Spec} \mathscr{O}_{X \times_S T,z}$. It is thus enough to show that these local obstructions vanish. Hence our situation is as follows: We have a surjection of local Artinian rings $R' \longrightarrow R$ whose kernel is 1-dimensional over the residue field, an essentially smooth 2-dimensional local R'-algebra S', and a quotient $S := S' \bigotimes_{R'} R \longrightarrow T$ such that T is a finite flat R-module. We then want to lift T to a quotient $S' \longrightarrow T'$ which is a flat R'-module. We first claim that T has projective dimension 2 over S. As T is R-flat it is enough to check $\overline{T}$ has projective dimension 2 over $\overline{S}$, where $\overline{(-)}$ denotes reduction modulo the maximal ideal of R. In that case we have that $\overline{T}$ is a Cohen-Macaulay module over the regular local ring $\overline{S}$ with support of codimension 2 and the result follows.

By [17, Thm. 7.15] (cf. also the original proof in [6]) it then follows that the ideal I_T defining T is the determinant ideal of $n \times n$-minors of an $n+1 \times n$-matrix M and that the grade (the maximal length of S-regular sequence contained in I_T) of I_T is 2. We then (arbitrarily) lift M to a matrix M' over S' and let T' be defined by $n \times n$-minors of M'. What remains to show is that T' is R'-flat. The grade of $I_{T'}$ is also 2 as we may lift an S-regular sequence in I_T to elements of $I_{T'}$ which then given an S'-regular sequence and hence by [17, Thm. 7.16], the sequence

$$0 \longrightarrow (S')^n \longrightarrow (S')^{n+1} \longrightarrow S' \longrightarrow T' \longrightarrow 0$$

is exact, where $(S')^n \longrightarrow (S')^{n+1}$ is given by the lifted matrix and $(S')^{n+1} \longrightarrow S'$ by its minors (with appropriate signs). For the same reason this sequence tensored with the residue field of R' remains exact which shows that T' is R'-flat. $\square$

Corollary 7.27. *Let $X \longrightarrow S$ be a smooth, separated morphism of pure relative dimension 2. Then we have that the Hilbert scheme $\mathrm{Hilb}^n_{X/S}$ is the blow-up of $\Gamma^n_{X/S}$ along Δ_X.*

Proof. If we can prove that U^{et} of $\mathrm{Hilb}^n_{X/S}$ is schematically dense then we are finished by the Theorem. As the defining ideal of the complement of U^{et} is locally principal and as $\mathrm{Hilb}^n_{X/S} \longrightarrow S$ is flat by the proposition this can be checked fibre by fibre so we may assume that S is the spectrum of a field k. Now, in that case $\mathrm{Hilb}^n_{X/S}$ is smooth by the proposition or by Fogarty's result. For the density statement we may reduce to the base field k being algebraically closed. Write $X = \sqcup_{i=1,\dots,p} X_i$ as a disjoint union. We have that $\mathrm{Hilb}^n_{X/S}$ is the disjoint union $\sqcup_{n_1+\cdots+n_p=n}\mathrm{Hilb}^{n_i}(X_i)$. As U^{et} is non-empty in each of the components $\mathrm{Hilb}^{n_i}(X_i)$, this implies that it is schematically dense in $\mathrm{Hilb}^n_{X/S}$. □

Remark 7.28. As pointed out by the referee, there is a small inaccuracy in ([10, Proposition 2.3]) concerning the connectedness of the Hilbert scheme in that the Hilbert scheme of a connected scheme is not necessarily connected. The proof had to take that into account.

8. The good component for affine varieties

We will in this last section generalize the approach Haiman gives in [13], using the fact that the Hilbert scheme Hilb^n_Y, for a projective scheme Y, can be embedded as a closed subscheme of the Grassmannian of rank n-quotients of $H^0(Y, \mathscr{O}_Y(N))$, when N is large enough. To simplify we assume that our base $\mathrm{Spec}(A)$ is Noetherian.

Proposition 8.1. *Let $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ be a finite type morphism of affine schemes, and let $V \subseteq F$ be an n-sufficiently big A-submodule. Let I_V and I_F be the ideals of norms associated to V and F, respectively. The natural morphism $\oplus_{m\geq 0} I_V^m \to \oplus_{m\geq 0} I_F^m$ induces a morphism*

$$\varphi\colon \mathrm{G}^n_{X/S} = \mathrm{Proj}(\oplus_{m\geq 0} I_F^m) \longrightarrow \mathrm{Bl}_{I_V}(\Gamma^n_A F) = \mathrm{Proj}(\oplus_{m\geq 0} I_V^m)$$

which is finite.

Proof. Let U resp. U' be the complement of $\mathrm{Spec}(\Gamma^n_A F)$ in $\mathrm{Spec}(\oplus_{m\geq 0} I_F^m)$ resp. $\mathrm{Spec}(\oplus_{m\geq 0} I_V^m)$. That the map on Proj's is well-defined means that the map on spectra maps U into U'. Assume therefore, by way of contradiction, that we have a closed point x of U that does not map into U'. This gives us a field valued point of $\mathrm{Hilb}^n_{\mathrm{Spec}(F)/\mathrm{Spec}(A)}$, i.e., an n-dimensional quotient $F \bigotimes_A k \to R$. However, the assumption that the image of x does not lie in U' means that the image of V does not span R. This however contradicts the assumption that V is n-sufficiently big.

For graded elements f in a graded ring R we let $D_+(f)$ denote the basic open affine given as the spectrum of the degree zero part of the localized ring R_f. We have, for any $f \in I_V$ that $\varphi^{-1}(D_+(f)) = D_+(f)$, hence the morphism φ is an affine morphism. Since F is assumed

of finite type it follows from Lemma (2.8) that I_F is of finite type, and consequently $\mathrm{G}^n_{X/S}$ is proper over $\mathrm{Spec}(\Gamma^n_A F)$. Since $\mathrm{Bl}_{I_V}(\Gamma^n_A F)$ is separated it follows that φ is proper. Thus the morphism φ is both proper and affine, hence finite. $\square$

When $V \subseteq F$ is n-sufficiently big we have an induced morphism

$$h\colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Grass}^n_V$$

from the Hilbert scheme to the Grassmannian.

Lemma 8.2. *Let $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ be of finite type, and let $V \subset F$ be n-sufficiently big, finitely generated A-module. We have a commutative diagram*

$$\begin{tikzcd} \mathrm{G}^n_{X/S} \arrow[r] \arrow[d, "\varphi"] & \mathrm{Hilb}^n_{X/S} \arrow[d, "h"] \\ \mathrm{Bl}_{I_V}(\Gamma^n_A F) \arrow[r] & \mathrm{Grass}^n_V \end{tikzcd}.$$

Proof. Since V is finitely generated we can use the Plücker coordinates to embed Grass^n_V as a closed subscheme of $\mathbf{P}(\wedge^n V)$. Composed with the diagonal embedding and the Segre embedding yields the closed immersion ι_1 given as the composite

$$\mathrm{Grass}^n_V \subset \mathbf{P}(\wedge^n V) \subset \mathbf{P}(\wedge^n V) \times \mathbf{P}(\wedge^n V) \subset \mathbf{P}(\wedge^n V \otimes \wedge^n V).$$

The natural map of A-modules $\wedge^n V \otimes_A \wedge^n V \longrightarrow I_V$ will by definition hit all the generators for the ideal I_V, and consequently determine a closed immersion $\iota_2\colon \mathrm{Bl}_{I_V}(\Gamma^n_A F) \longrightarrow \mathbf{P}(\wedge^n_V \otimes \wedge^n V) \times \mathrm{Spec}(\Gamma^n_A(F))$. We now have the commutative diagram

$$\begin{tikzcd} \mathrm{G}^n_{X/S} \arrow[r] \arrow[d, "\varphi"] & \mathrm{Hilb}^n_{X/S} \arrow[r, "h"] & \mathrm{Grass}^n_V \arrow[d, "\iota_1"] \\ \mathrm{Bl}_{I_V}(\Gamma^n_A F) \arrow[rr, "p_1 \circ \iota_2"] & & \mathbf{P}(\wedge^n V \otimes_A \wedge^n V) \end{tikzcd},$$

where p_1 is the projection on the first factor. The inverse image $\varphi^{-1}(E)$ of the exceptional divisor $E \subseteq \mathrm{Bl}_{I_V}(\Gamma^n_A F)$ is the exceptional divisor of $\mathrm{G}^n_{X/S}$, and on the open complement we have that φ is an isomorphism. Consequently $p_1 \circ \iota_2\colon \mathrm{Bl}_{I_V}(\Gamma^n_A F) \longrightarrow \mathbf{P}(\wedge^n V \otimes_A \wedge^n V)$ factors through Grass^n_V since it does so on the complement of a Cartier divisor. $\square$

8.3. Consider now $Y = \mathbf{P}^r_S$, and let $g\colon Y \longrightarrow S$ denote the structure map. For any closed subscheme $Z \subseteq Y$ that is flat, locally free of rank n over S, the induced map

$$g_*\mathscr{O}_Y(N) \longrightarrow g_*\mathscr{O}_Z(N) \tag{8.3.1}$$

is easily seen to be surjective for $N \geq n-1$. Furthermore, the ideal sheaf $\mathscr{I}_Z$ twisted with $N \geq n$ is regular, that is $R^p g_* \mathscr{I}_Z(N-p) = 0$ for $p > 0$ when $N \geq n$. It follows ([11]) that the induced morphism

$$\mathrm{Hilb}^n_{Y/S} \longrightarrow \mathrm{Grass}^n_{g_* \mathscr{O}_Y(N)} \tag{8.3.2}$$

is a closed immersion for $N \geq n$.

Proposition 8.4. *Let F be an A-algebra generated by $t_1, \dots, t_r$, let $V \subseteq F$ be spanned by the monomials of degree $\leq n$ in the $t_1, \dots, t_r$. Then the morphism*

$$\varphi\colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}_{I_V}(\Gamma^n_A F)$$

is an isomorphism.

Proof. We embed $X = \mathrm{Spec}(F)$ in $Y = \mathbf{P}^r_S$ using $(1\colon t_1\colon \cdots : t_r)$. The natural map $h\colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Grass}^n_V$ is an immersion and the natural map $\mathrm{Grass}^n_V \to \mathrm{Grass}^n_{g_*(\mathscr{O}_Y(N))}$ is a closed immersion. As $\mathrm{Hilb}^n_{X/S}$ immerses into $\mathrm{Hilb}^n_{Y/S}$, and the map (8.3.2) is an immersion, it follows that the map $h\colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Grass}^n_V$ is an immersion.

By Lemma (8.2) we have that the restriction of h to $\mathrm{G}^n_{X/S}$ factors through

$$\varphi\colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}_{I_V}(\Gamma^n_A F),$$

hence φ must be an immersion as well. However, by Proposition (8.1) the map φ is proper, and consequently we have that the map φ must be a closed immersion. Furthermore, since φ is an isomorphism over the complement of a Cartier divisor, it is an isomorphism. □

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